Asymptotics for the low-lying eigenstates of the Schroedinger operator with magnetic field near corners

Virginie Bonnaillie-Noël, Monique Dauge

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Abstract

The Neumann realization for the Schrödinger operator with magnetic field is considered in a bounded two-dimensional domain with corners. This operator is associated with a small semi-classical parameter h or, equivalently, with a large magnetic field. We investigate the behavior of its eigenpairs as h tends to zero, like in a semi-classical limit. We prove, in the situation where the domain is a polygon and the magnetic field is constant, that the lowest eigenvalues are exponentially close to those of model problems associated with the corners. We approximate the corresponding eigenvectors by linear combinations of functions concentrated in corners at the scale $\sqrt{h}$. If the domain has curved sides and the magnetic field is smoothly varying, we exhibit a full asymptotics for eigenpairs in powers of $\sqrt{h}$.

Contents

1 Introduction

The topic of our paper takes its origin from the Ginzburg-Landau theory modeling superconducting properties in presence of an external magnetic field [11, 28]: The study of the Hessian of the Ginzburg-Landau functional leads to analyze the ground state of the Schrödinger operator with magnetic field [12, 18]. A small semi-classical parameter $h = (\kappa\mathcal{B})^{-1}$ appears as the magnetic field $\mathcal{B}$ is large or the physical characteristic κ of the superconducting material is large. When compared with most of the literature about Schrödinger operators, the unusual feature of the resulting problem is that it is posed on *subdomains* of $\mathbb{R}^2$ or $\mathbb{R}^3$, and subject to Neumann or impedance boundary conditions.

Motivated by this, and also by other works about the spectrum of Schrödinger operator in the semi-classical limit, see [19, 20] for instance, we deal with the asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field in a bounded two-dimensional domain, with focus on the influence of *convex corners*.

Let Ω denote a bounded domain in $\mathbb{R}^2$ and $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ a smooth magnetic potential associated with its magnetic field $\mathcal{B} = \operatorname{curl}\mathcal{A}$. It is assumed that $\mathcal{B} > 0$ on $\overline{\Omega}$. We investigate the behavior of the eigenpairs of the Neumann realization P_h on Ω for the Schrödinger operator $-(h\nabla - i\mathcal{A})^2$ as $h \to 0$.

Many papers are devoted to the analysis of the first eigenpair when Ω is a smooth domain. We can quote works of Bernoff-Sternberg [3], Lu-Pan [22, 23], Helffer-Morame [16, 17]: It is proved that the fundamental state is localized near points of the boundary where the curvature is maximal, and a two-term asymptotics of the fundamental state energy of P_h is given. More recently, Fournais-Helffer [14] prove a complete asymptotic expansion for low-lying eigenvalues of P_h on domains such that the boundary curvature reaches its maximum in only one point.

Although the interest for non smooth domain is often mentioned in physical literature [8, 13, 26], quite few mathematical papers are devoted to that problem: Let us mention contributions of Jadallah [21], Pan [24] which deal with very particular domains like a square or a quarter plane. More recently, [5] gives a systematic analysis for infinite sectors of $\mathbb{R}^2$, proving an asymptotics of the smallest eigenvalue of $-(\nabla - i\mathcal{A})^2$ when the *aperture* α *of the sector tends to* 0, and exponential decay estimates for the corresponding eigenfunctions. The limit as $h \to 0$ of the first eigenvalue of P_h for domains with corners is deduced.

In this paper, we prove sharper results, exhibiting the complete asymptotic expansion of low-lying eigenstates for curvilinear polygonal domains. We also prove refined results in the case when the domain has straight sides and the magnetic field is constant: The convergence of the eigenpairs to their limits is then exponential, behaving as $\exp(-\beta/\sqrt{h})$ for a positive β depending on their rank.

2 Outline

Let us sketch our results. The behavior of the first eigenstates of P_h depends on the spectrum of *model problems* associated with each point of the boundary, in particular, those associated with the corners s of Ω. Section 3 is devoted to spectral and solvability properties of such model operator $Q^\alpha := -(\nabla - i\mathcal{A}_0)^2$ on an infinite sector of opening α and vertex at the origin. Here $\mathcal{A}_0$ is the canonical magnetic potential $\frac{1}{2}(-\mathsf{X}_2, \mathsf{X}_1)$ corresponding to the magnetic field $\mathcal{B} = 1$. For any opening α, the essential spectrum of the operator Q^α is equal to $[\Theta_0, +\infty)$, with the universal constant $\Theta_0 \simeq 0.590125$. Depending on the value of α, the discrete spectrum of Q^α is empty or consists of K_α eigenvalues. The corresponding eigenvectors are exponentially decreasing and, moreover, solutions Ψ of $Q^\alpha\Psi = \mathcal{L}$ with Neumann conditions and exponentially decreasing right hand side $\mathcal{L}$, are exponentially decreasing, too.

Sections 4-5 are devoted to the Schrödinger operator P_h when the domain Ω is a polygon, i.e. its sides are segments on lines, and the magnetic field $\mathcal{B}$ is equal to 1. To fix ideas, the magnetic potential is taken as $\frac{1}{2}(-x_2, x_1)$. The eigenvectors of the model operators $Q^{\alpha_{\mathsf{s}}}$ corresponding to the aperture α_{s} at each corner s of Ω allow the construction of quasi-modes in Section 4. These quasi-modes generate a space of dimension $K_\Omega := \sum_{\mathsf{s}} K_{\alpha_{\mathsf{s}}}$, the sum of the contributions of each corner. In Section 5, we prove that the first K_Ω eigenvalues of P_h, when divided by h, converge exponentially fast towards the eigenvalues of the model operators $\oplus_{\mathsf{s}} Q^{\alpha_{\mathsf{s}}}$. We also prove the localization of their eigenfunctions in corresponding corners. Let us emphasize that, when several corners have the same aperture, clustering of eigenvalues appear, and that each of the corresponding eigenvectors may concentrate in the vicinity of several corners.

In Sections 6-7, we analyze more general domains (curvilinear polygons) with smoothly varying magnetic fields $\mathcal{B}$. Again, we use the model operators $Q^{\alpha_{\mathsf{s}}}$ to construct quasi-modes for P_h, but now in combination with a formal series calculus. We obtain asymptotics series in powers of $\sqrt{h}$ for a finite number of low-lying eigenstates of P_h. In Section 8, we conclude our paper by commenting on numerical approximation issues: The eigenmodes have a two-scale structure, in the form of the product of a corner layer at scale $\sqrt{h}$ with an oscillatory term at scale h. The latter makes the numerical approximation delicate, see [1, 2, 7]. A finite element method using high degree polynomials is being investigated by the authors, together with the tunneling effect in presence of symmetries.

3 Model operators in infinite sectors

The model problem associated with a corner of opening α in the domain Ω is a Schrödinger operator Q^α in an infinite sector G^α of same opening, with a model magnetic potential $\mathcal{A}_0$ corresponding to a constant field equal to 1. After recalling results from [5] on the spectrum of this operator, we study its solvability in spaces of exponentially decreasing functions. We end this section by stating the relation between this model problem and a more general Schrödinger operator $Q^{\alpha,\mathcal{A}}$ associated with any affine magnetic potential $\mathcal{A}$.

3.1 Spectrum

We denote by $\mathsf{X} = (\mathsf{X}_1, \mathsf{X}_2)$ the Cartesian coordinates in $\mathbb{R}^2$, and by $R = |\mathsf{X}|$ and θ the polar coordinates. Let G^α be the sector in $\mathbb{R}^2$ with opening α:

$$G^\alpha = \{\mathsf{X} \in \mathbb{R}^2, \quad \theta \in (0, \alpha)\}.$$

We consider the model magnetic potential $\mathcal{A}_0$ defined on $\mathbb{R}^2$ by

$$\mathcal{A}_0(\mathsf{X}) = \frac{1}{2}(-\mathsf{X}_2, \mathsf{X}_1). \tag{1}$$

Then the magnetic field $\mathcal{B}$ given by $\operatorname{curl}\mathcal{A}_0$ is equal to 1. Let Q^α be the Neumann realization of the Schrödinger operator $-(\nabla - i\mathcal{A}_0)^2$ on the sector G^α. The operator Q^α is associated with the sesquilinear form q^α defined on the variational space $\mathcal{V}(q^\alpha)$ as follows:

$$\mathcal{V}(q^\alpha) \;=\; \Big\{\Psi \in L^2(G^\alpha),\;\; (\nabla - i\mathcal{A}_0)\Psi \in L^2(G^\alpha)\Big\}, \tag{2}$$

$$q^\alpha(\Psi, \Phi) \;=\; \int_{G^\alpha} (\nabla - i\mathcal{A}_0)\Psi(\mathsf{X}) \cdot \overline{(\nabla - i\mathcal{A}_0)\Phi(\mathsf{X})}\, d\mathsf{X}, \;\; \Psi, \Phi \in \mathcal{V}(q^\alpha). \tag{3}$$

The norm attached with the space $\mathcal{V}(q^\alpha)$ is

$$||\Psi||^2_{\mathcal{V}(q^\alpha)} = ||\Psi||^2_{L^2(G^\alpha)} + ||(\nabla - i\mathcal{A}_0)\Psi||^2_{L^2(G^\alpha)}.$$

Note that if $\Psi \in \mathcal{V}(q^\alpha)$, then for any ball B, $\Psi \in H^1(G^\alpha \cap B)$. Conversely, any Ψ in $L^2(G^\alpha)$ such that $\nabla\Psi$ and $|X|\Psi$ are in $L^2(G^\alpha)$, belongs to $\mathcal{V}(q^\alpha)$.

Then the operator Q^α associated with the form q^α writes

$$Q^\alpha = -(\nabla - i\mathcal{A}_0)^2 = -\Delta + i(\mathsf{X}_1\partial_{\mathsf{X}_2} - \mathsf{X}_2\partial_{\mathsf{X}_1}) + \frac{1}{4}|\mathsf{X}|^2. \tag{4}$$

It is defined on its domain $\mathcal{D}(Q^\alpha)$:

$$\mathcal{D}(Q^\alpha) = \Big\{\Psi \in \mathcal{V}(q^\alpha),\ \ (\nabla - i\mathcal{A}_0)^2\Psi \in L^2(G^\alpha),\ \ \nu\cdot(\nabla - i\mathcal{A}_0)\Psi\big|_{\partial G^\alpha} = 0\Big\}.$$

Here ν is the outward unit normal on the boundary of G^α.

The operator Q^α is hermitian and positive. The lowest part of its spectrum can be defined by Rayleigh quotients.

Definition 3.1. *Let $\mu_k(\alpha)$ be the $k-th$ smallest element of the spectrum of Q^α, given by the max-min principle:*

$$\mu_k(\alpha) = \max_{\Psi_1,\ldots,\Psi_{k-1}} \min\left\{\frac{q^\alpha(\Psi,\Psi)}{\langle\Psi,\Psi\rangle},\ \ \Psi \in \mathcal{V}(q^\alpha),\ \ \Psi \in [\Psi_1,\ldots,\Psi_{k-1}]^\perp\right\}. \tag{5}$$

Here $\langle\cdot,\cdot\rangle$ denotes the hermitian scalar product of $L^2(G^\alpha)$.

Let us quote some results of [5] about the spectrum of Q^α.

Theorem 3.2.

(i) The infimum of the essential spectrum of Q^α is equal to $\Theta_0 := \mu_1(\pi)$.

(ii) For all $\alpha \in (0, \frac{\pi}{2}]$, $\mu_1(\alpha) < \Theta_0$ and, therefore, $\mu_1(\alpha)$ is an eigenvalue.

(iii) Let k be a positive integer and $\alpha > 0$ such that $\mu_k(\alpha) < \Theta_0$. We denote by Ψ_k^α a normalized eigenfunction associated with $\mu_k(\alpha)$. Then Ψ_k^α satisfies the following exponential decay estimate:

$$\forall\varepsilon > 0, \exists C_{\varepsilon,\alpha}, \quad \left\|\mathrm{e}^{\left(\sqrt{\Theta_0-\mu_k(\alpha)}-\varepsilon\right)|\mathsf{X}|}\,\Psi_k^\alpha\right\|_{\mathcal{V}(q^\alpha)} \le C_{\varepsilon,\alpha}. \tag{6}$$

Remark 3.3. Based on the asymptotics of $\mu_1(\alpha)$ as $\alpha \to 0$, see [5], and numerical computations, see [4, 1], we conjecture that μ_1 is increasing from $(0,\pi]$ onto $(0,\Theta_0]$ and equal to Θ_0 on $[\pi, 2\pi)$.

3.2 Solvability and exponential decay

We firstly prove the Fredholm alternative for the operator $Q^\alpha - \mu\mathrm{Id}$, if μ is an eigenvalue. Then, we prove the exponential decay of solutions if the right hand side is itself exponentially decaying. We recall notation partially introduced in Theorem 3.2.

Notation 3.4.
Let $\alpha \in (0, 2\pi)$ and K_α be the largest integer such that $\mu_{K_\alpha}(\alpha) < \Theta_0$.
- We denote by $(\Psi_j^\alpha)_{1\leq j\leq K_\alpha}$ an orthonormalized system of eigenfunctions respectively associated with $\mu_j(\alpha)$ for the operator Q^α.
- Let $k \in \mathbb{N}$ with $k = 1$, or $2 \leq k \leq K_\alpha$ and such that $\mu_{k-1}(\alpha) < \mu_k(\alpha)$. Let l be the multiplicity of $\mu_k(\alpha)$. Thus, we have

$$\mu_{k-1}(\alpha) < \mu_k(\alpha) = \ldots = \mu_{k+l-1}(\alpha) < \mu_{k+l}(\alpha) \leq \Theta_0.$$

Lemma 3.5. *With Notation* 3.4*, let $\mathcal{L}$ be a linear form defined and continuous on $\mathcal{V}(q^\alpha)$, and such that*

$$\mathcal{L}(\Psi_j^\alpha) = 0, \quad \forall j = k, \ldots, k+l-1. \tag{7}$$

Then, there exists a unique $\Psi \in \mathcal{V}(q^\alpha)$ such that

$$\begin{cases} \langle \Psi, \Psi_j^\alpha \rangle = 0, \quad \forall j = k, \ldots, k+l-1, \\ q^\alpha(\Psi, \Phi) - \mu_k(\alpha)\langle \Psi, \Phi\rangle = \mathcal{L}(\Phi), \quad \forall \Phi \in \mathcal{V}(q^\alpha), \end{cases} \tag{8}$$

with q^α defined by (3) *and $\langle \cdot, \cdot \rangle$ the L^2-scalar product on G^α. If we assume that, moreover, $\mathcal{L}(\Psi_j^\alpha) = 0$ for all $j = 1, \ldots, k-1$, the solution of* (8) *is orthogonal to $\Psi_1^\alpha, \ldots, \Psi_{k-1}^\alpha$.*

Proof. Let $N \geq k+l-1$ such that $\mu_N(\alpha) < \Theta_0$. With Notation 3.4, it is enough to choose $k+l-1 \leq N \leq K_\alpha$. We split the linear form $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_0 + \sum_{j=1}^{k-1} c_j \Psi_j^\alpha + \sum_{j=k+l}^{N} c_j \Psi_j^\alpha \quad \text{with} \quad \mathcal{L}_0(\Psi_j^\alpha) = 0, \ \forall j = 1, \ldots, N.$$

We define the space

$$\mathcal{V}^N = \{\Psi \in \mathcal{V}(q^\alpha), \quad \langle \Psi, \Psi_j^\alpha \rangle = 0, \ \forall j = 1, \ldots, N\}.$$

Let us prove that the sesquilinear form $q^\alpha - \mu_k(\alpha)\langle \cdot, \cdot \rangle$ is coercive on $\mathcal{V}^N$: Let $\kappa \in (0,1)$ and $\Psi \in \mathcal{V}^N$, then

$$\begin{aligned} q^\alpha(\Psi,\Psi) - \mu_k(\alpha)\langle\Psi,\Psi\rangle &\geq (1-\kappa)q^\alpha(\Psi,\Psi) + (\kappa\mu_{N+1}(\alpha) - \mu_k(\alpha))\langle\Psi,\Psi\rangle \\ &\geq \min(1-\kappa, \kappa\mu_{N+1}(\alpha) - \mu_k(\alpha))\,||\Psi||^2_{\mathcal{V}(q^\alpha)}. \end{aligned}$$

It suffices to choose $\kappa \in (0,1)$ such that $\kappa\mu_{N+1}(\alpha) - \mu_k(\alpha) > 0$ to deduce the coercivity.

Therefore, by the Lax-Milgram theorem, there exists a unique $\Psi_0 \in \mathcal{V}^N$ such that

$$q^\alpha(\Psi_0, \Phi) - \mu_k(\alpha)\langle \Psi_0, \Phi\rangle = \mathcal{L}_0(\Phi), \quad \forall \Phi \in \mathcal{V}^N.$$

By orthogonality, Ψ_0 is the unique solution in $\mathcal{V}(q^\alpha)$ for the problem

$$\left\{\begin{array}{l}\langle\Psi_0,\Psi_j^\alpha\rangle = 0, \quad \forall j=1,\dots,N,\\ q^\alpha(\Psi_0,\Phi)-\mu_k(\alpha)\langle\Psi_0,\Phi\rangle = \mathcal{L}_0(\Phi), \quad \forall\Phi\in\mathcal{V}(q^\alpha).\end{array}\right. \tag{9}$$

Furthermore, for $j\in\{1,\dots,k-1,k+l,\dots,N\}$, the unique function $u_j\in\mathcal{V}(q^\alpha)$ orthogonal to $\Psi_k^\alpha,\dots,\Psi_{k+l-1}^\alpha$ such that $(Q^\alpha-\mu_k(\alpha))u_j=\Psi_j^\alpha$ is given by

$$u_j = \frac{1}{\mu_j(\alpha)-\mu_k(\alpha)}\ \Psi_j^\alpha.$$

Consequently,

$$\Psi := \Psi_0 + \sum_{j=1}^{k-1}\frac{c_j}{\mu_j(\alpha)-\mu_k(\alpha)}\ \Psi_j^\alpha + \sum_{j=k+l}^{N}\frac{c_j}{\mu_j(\alpha)-\mu_k(\alpha)}\ \Psi_j^\alpha \tag{10}$$

is the unique solution of (8). $\square$

Let us now analyze the decay of the solution Ψ as split in (10). Theorem 3.2 gives the decay of Ψ_j^α. Therefore, it is enough to study the decay of Ψ_0.

Lemma 3.6. *With Notation* 3.4*, let N be an integer, $k+l-1\le N\le K_\alpha$. Let $\mathcal{L}_0$ be a linear form continuous on $\mathcal{V}(q^\alpha)$. We assume*

$$\mathcal{L}_0(\Psi_j^\alpha)=0, \quad \forall j=1,\dots,N, \tag{11}$$

and that, moreover, there exists $\delta_0>0$ such that $\mathcal{L}_0$ is defined on $\{\mathrm{e}^{\delta_0|\mathsf{X}|}\Psi,\ \Psi\in\mathcal{V}(q^\alpha)\}$ with the estimate

$$\exists C>0, \quad \forall\Phi\in\mathcal{V}(q^\alpha), \qquad \left|\mathcal{L}_0(\mathrm{e}^{\delta_0|\mathsf{X}|}\Phi)\right| \le C||\Phi||_{\mathcal{V}(q^\alpha)}. \tag{12}$$

Then, the solution $\Psi_0\in\mathcal{V}(q^\alpha)$ of (9) *satisfies $\mathrm{e}^{\delta_N|\mathsf{X}|}\Psi_0\in\mathcal{V}(q^\alpha)$ for some positive number $\delta_N\le\delta_0$, independent of $\mathcal{L}_0$.*

Proof. Let $\delta\le\delta_0$. We define $\Psi_\delta=\mathrm{e}^{\delta|\mathsf{X}|}\Psi_0$, and we check that for any $\Phi\in\mathcal{V}(q^\alpha)$,

$$\begin{aligned} q^\alpha(\Psi_0,\Phi) &= q^\alpha(\mathrm{e}^{-\delta|\mathsf{X}|}\Psi_\delta,\Phi)\\ &= \int_{G^\alpha}(\nabla-i\mathcal{A}_0-\delta\mathcal{I})\Psi_\delta\cdot\overline{(\nabla-i\mathcal{A}_0+\delta\mathcal{I})(\mathrm{e}^{-\delta|\mathsf{X}|}\Phi)}\,d\mathsf{X},\end{aligned}$$

with $\mathcal{I}=\begin{pmatrix}1\\1\end{pmatrix}$. Let us define the space $\mathcal{V}_\delta^N$ and the form a_δ on $\mathcal{V}_\delta^N\times\mathcal{V}_\delta^N$ by

$$\begin{aligned}\mathcal{V}_\delta^N &= \left\{\Psi\in\mathcal{V}(q^\alpha), \quad \langle\Psi,\mathrm{e}^{-\delta|\mathsf{X}|}\Psi_j^\alpha\rangle=0,\ \ \forall j=1,\dots,N\right\},\\ a_\delta(\Psi,\Phi) &= \int_{G^\alpha}\Big((\nabla-i\mathcal{A}_0-\delta\mathcal{I})\Psi\cdot\overline{(\nabla-i\mathcal{A}_0+\delta\mathcal{I})\Phi}-\mu_k(\alpha)\Psi\,\overline{\Phi}\Big)\,d\mathsf{X}.\end{aligned}$$

Then, for any $\Phi \in \mathcal{V}(q^\alpha)$, we have

$$\mathcal{L}_0(\Phi) = q^\alpha(\Psi_0, \Phi) - \mu_k(\alpha)\langle \Psi_0, \Phi\rangle = a_\delta(\Psi_\delta, \mathrm{e}^{-\delta|\mathsf{X}|}\Phi) = \mathcal{L}_\delta(\mathrm{e}^{-\delta|\mathsf{X}|}\Phi), \qquad (13)$$

where, thanks to (12), the linear form $\mathcal{L}_\delta$ can be defined on $\mathcal{V}(q^\alpha)$ by

$$\mathcal{L}_\delta(\Phi) = \mathcal{L}_0(\mathrm{e}^{\delta|\mathsf{X}|}\Phi). \qquad (14)$$

Due to the compatibility condition (11), we have

$$\mathcal{L}_\delta(\mathrm{e}^{-\delta|\mathsf{X}|}\Psi_j^\alpha) = 0, \quad \forall j = 1, \dots, N.$$

Solving the problem of finding $\Phi_\delta \in \mathcal{V}_\delta^N$ such that

$$a_\delta(\Phi_\delta, \Phi) = \mathcal{L}_\delta(\Phi), \quad \forall \Phi \in \mathcal{V}_\delta^N, \qquad (15)$$

will provide exponential decay of the solution Ψ_0 for problem (8).

We verify easily that the form a_δ is sesquilinear and continuous on $\mathcal{V}_\delta^N \times \mathcal{V}_\delta^N$. Let us prove its coercivity. Let $\Psi \in \mathcal{V}_\delta^N$, then

$$|a_\delta(\Psi, \Psi)| \geq \text{ Re } a_\delta(\Psi, \Psi) = q^\alpha(\Psi, \Psi) - \mu_k(\alpha)||\Psi||^2 - \delta^2||\Psi||^2. \qquad (16)$$

We decompose Ψ such that

$$\Psi = \sum_{j=1}^N \langle \Psi, \Psi_j^\alpha\rangle \Psi_j^\alpha + \Psi^\perp,$$

then, the definition of $\mathcal{V}_\delta^N$ and the decay of Ψ_j^α give for any $j = 1, \dots, N$,

$$\begin{aligned} \left|\langle \Psi, \Psi_j^\alpha\rangle\right| &= \left|\langle \Psi, (1 - \mathrm{e}^{-\delta|\mathsf{X}|})\Psi_j^\alpha\rangle\right| \\ &\leq ||\Psi||_{L^2(G^\alpha)} \left(\int_{G^\alpha} \delta^2|\mathsf{X}|^2|\Psi_j^\alpha(\mathsf{X})|^2 \, d\mathsf{X}\right)^{1/2} \\ &\leq C_j\, \delta ||\Psi||_{L^2(G^\alpha)}. \end{aligned} \qquad (17)$$

Furthermore, due to the decomposition of Ψ and (17), it follows

$$\begin{aligned} q^\alpha(\Psi, \Psi) &= q^\alpha(\Psi^\perp, \Psi^\perp) + \sum_{j=1}^N \left|\langle \Psi, \Psi_j^\alpha\rangle\right|^2 q^\alpha(\Psi_j^\alpha, \Psi_j^\alpha) \\ &\geq \mu_{N+1}(\alpha)||\Psi^\perp||^2 + \sum_{j=1}^N \mu_j(\alpha)\left|\langle \Psi, \Psi_j^\alpha\rangle\right|^2 \\ &\geq \mu_{N+1}(\alpha)||\Psi||^2 - \sum_{j=1}^N (\mu_{N+1}(\alpha) - \mu_j(\alpha))\left|\langle \Psi, \Psi_j^\alpha\rangle\right|^2 \\ &\geq \left((\mu_{N+1}(\alpha) - \delta^2 \sum_{j=1}^N C_j^2\,(\mu_{N+1}(\alpha) - \mu_j(\alpha))\right)||\Psi||^2. \end{aligned} \qquad (18)$$

Defining $M_N = \sum_{j=1}^{N} C_j^2(\mu_{N+1}(\alpha) - \mu_j(\alpha))$, we deduce from (16)-(18), for $\kappa \in (0,1)$,

$$|a_\delta(\Psi,\Psi)| \geq (1-\kappa)||(\nabla - i\mathcal{A}_0)\Psi||^2 + \Big(\kappa\big(\mu_{N+1}(\alpha) - \delta^2 M_N\big) - \mu_k(\alpha) - \delta^2\Big)||\Psi||^2.$$

We define

$$\bar{\delta}_N = \min\left(\delta_0\,, \sqrt{\frac{\mu_{N+1}(\alpha) - \mu_k(\alpha)}{1+M_N}}\right).$$

For $\delta \in (0, \bar{\delta}_N)$, we choose $\kappa \in (0,1)$ such that $\kappa\big(\mu_{N+1}(\alpha) - \delta^2 M_N\big) - \mu_k(\alpha) - \delta^2 > 0$. This choice proves the coercivity of a_δ on $\mathcal{V}_\delta^N$ for any $\delta \in [0, \bar{\delta}_N)$.

Applying the Lax-Milgram theorem, we find that there exists a unique $\Phi_\delta \in \mathcal{V}_\delta^N$ solution of the variational problem (15). Thanks to the orthogonality conditions, Φ_δ satisfies, moreover, $a_\delta(\Phi_\delta, \Phi) = \mathcal{L}_\delta(\Phi)$, $\forall \Phi \in \mathcal{V}(q^\alpha)$. Since δ is positive, Φ_δ satisfies, a fortiori,

$$a_\delta(\Phi_\delta, \mathrm{e}^{-\delta|\mathsf{X}|}\Phi) = \mathcal{L}_\delta(\mathrm{e}^{-\delta|\mathsf{X}|}\Phi), \quad \forall \Phi \in \mathcal{V}(q^\alpha).$$

Therefore Φ_δ coincides with $\Psi_\delta = \mathrm{e}^{\delta|\mathsf{X}|}\Psi_0$, compare with (13). We deduce that $\mathrm{e}^{\delta|\mathsf{X}|}\Psi_0$ belongs to $\mathcal{V}(q^\alpha)$ for all $\delta \in [0, \bar{\delta}_N)$, which ends the proof. □

Using Lemmas 3.5 and 3.6, we deduce:

Lemma 3.7. *With Notation* 3.4*, let $\mathcal{L}$ be a linear form continuous on $\mathcal{V}(q^\alpha)$. We assume*

$$\mathcal{L}(\Psi_j^\alpha) = 0, \quad \forall j = k, \dots, k+l-1, \tag{19}$$

and that, moreover, there exists $\delta_0 > 0$ such that $\mathcal{L}$ is defined on $\mathrm{e}^{\delta_0|\mathsf{X}|}\mathcal{V}(q^\alpha)$ with the estimate:

$$\exists C > 0, \quad \forall \Phi \in \mathcal{V}(q^\alpha), \quad \Big|\mathcal{L}(\mathrm{e}^{\delta_0|\mathsf{X}|}\Phi)\Big| \leq C||\Phi||_{\mathcal{V}(q^\alpha)}. \tag{20}$$

Then, there exists a unique $\Psi \in \mathcal{V}(q^\alpha)$ such that

$$\begin{cases} \langle \Psi, \Psi_j^\alpha \rangle = 0, \quad \forall j = k, \dots, k+l-1, \\ q^\alpha(\Psi, \Phi) - \mu_k(\alpha)\langle \Psi, \Phi \rangle = \mathcal{L}(\Phi), \quad \forall \Phi \in \mathcal{V}(q^\alpha). \end{cases} \tag{21}$$

Furthermore, $\mathrm{e}^{\delta|\mathsf{X}|}\Psi \in \mathcal{V}(q^\alpha)$ for some $\delta \leq \delta_0$ independent of $\mathcal{L}$.

Proof. The existence and uniqueness is clear according to Lemma 3.5. Combining Lemma 3.6 and Theorem 3.2 with the decomposition of Ψ into

$$\Psi_0 + \sum_{j=1}^{k-1} \frac{c_j}{\mu_j(\alpha) - \mu_k(\alpha)}\ \Psi_j^\alpha + \sum_{j=k+l}^{N} \frac{c_j}{\mu_j(\alpha) - \mu_k(\alpha)}\ \Psi_j^\alpha,$$

we obtain the decay of Ψ for any $\delta < \min\Big(\bar{\delta}_N, \sqrt{\Theta_0 - \mu_N(\alpha)}\Big)$, with any integer N such that $k+l-1 \leq N \leq K_\alpha$. □

3.3 Generalization to any affine magnetic potential

To conclude this section about model problems, we deal with an arbitrary real-valued affine magnetic potential, thus of the form

$$\mathcal{A}(\mathsf{X}) = \Big(a_{11}\mathsf{X}_1 + a_{12}\mathsf{X}_2 + a_{10}\,,\ a_{21}\mathsf{X}_1 + a_{22}\mathsf{X}_2 + a_{20}\Big). \tag{22}$$

The associated magnetic field is

$$\mathcal{B} = \operatorname{curl}\mathcal{A} = a_{21} - a_{12}. \tag{23}$$

The quadratic function (the gauge function) defined by

$$\mathcal{G}(\mathsf{X}) = \frac{1}{2}\Big(a_{11}\mathsf{X}_1^2 + a_{22}\mathsf{X}_2^2 + (a_{12}+a_{21})\mathsf{X}_1\mathsf{X}_2\Big) + a_{10}\mathsf{X}_1 + a_{20}\mathsf{X}_2, \tag{24}$$

is such that

$$\mathcal{A} = \mathcal{B}\mathcal{A}_0 + \nabla\mathcal{G}, \tag{25}$$

with the model magnetic potential $\mathcal{A}_0$ defined in (1).

Proposition 3.8. *Let $\alpha \in (0,2\pi)$, and let $\mathcal{A}$ be an affine magnetic potential as in* (22)-(25). *We assume that the associated magnetic field $\mathcal{B}$ is positive. Then the Neumann realization, $Q^{\alpha,\mathcal{A}}$, of the Schrödinger operator $-(\nabla - i\mathcal{A})^2$ on G^α has K_α eigenvalues strictly less than $\mathcal{B}\Theta_0$. For $k \le K_\alpha$, the k-th eigenvalue of $-(\nabla - i\mathcal{A})^2$ is equal to $\mathcal{B}\mu_k(\alpha)$ and its corresponding normalized eigenvector, $\Psi_k^{\alpha,\mathcal{A}}$, is given on G^α by*

$$\Psi_k^{\alpha,\mathcal{A}}(\mathsf{X}) = \sqrt{\mathcal{B}}\exp\left(i\,\mathcal{G}(\mathsf{X})\right)\ \Psi_k^\alpha(\sqrt{\mathcal{B}}\,\mathsf{X}).$$

Proof. We verify easily that $\Psi_k^{\alpha,\mathcal{A}}$ is L^2-normalized. The operator $Q^{\alpha,\mathcal{A}}$ is defined on $\mathcal{D}(Q^{\alpha,\mathcal{A}})$ with

$$\begin{aligned}\mathcal{D}(Q^{\alpha,\mathcal{A}}) = \big\{\Psi \in L^2(G^\alpha),\ &(\nabla - i\mathcal{A})\Psi \in L^2(G^\alpha),\\ &(\nabla - i\mathcal{A})^2\Psi \in L^2(G^\alpha), \nu\cdot(\nabla - i\mathcal{A})\Psi_{|\partial G^\alpha} = 0\big\}.\end{aligned} \tag{26}$$

Using the transformation

$$\begin{array}{rcl}\mathcal{D}(Q^\alpha) & \to & \mathcal{D}(Q^{\alpha,\mathcal{A}})\\ \Psi & \mapsto & \Psi^{\mathcal{A}} \text{ with } \Psi^{\mathcal{A}}(\mathsf{X}) = \sqrt{\mathcal{B}}\exp(i\,\mathcal{G}(\mathsf{X}))\ \Psi(\sqrt{\mathcal{B}}\,\mathsf{X}),\end{array}$$

we see that the change of variables $\mathsf{Y} = \sqrt{\mathcal{B}}\,\mathsf{X}$ leads to

$$\begin{aligned}(\nabla_{\mathsf{X}} - i\mathcal{A}(\mathsf{X}))\Psi^{\mathcal{A}}(\mathsf{X}) &= \sqrt{\mathcal{B}}\exp(i\,\mathcal{G}(\mathsf{X}))\Big(\sqrt{\mathcal{B}}\,\nabla_{\mathsf{Y}} - i(\mathcal{A}(\mathsf{X}) - \nabla\mathcal{G}(\mathsf{X}))\Big)\Psi(\mathsf{Y})\\ &= \sqrt{\mathcal{B}}\exp(i\,\mathcal{G}(\mathsf{X}))\left(\sqrt{\mathcal{B}}\,\nabla_{\mathsf{Y}} - i\mathcal{B}\,\mathcal{A}_0\left(\frac{\mathsf{Y}}{\sqrt{\mathcal{B}}}\right)\right)\Psi(\mathsf{Y})\\ &= \mathcal{B}\exp(i\,\mathcal{G}(\mathsf{X}))\ (\nabla_{\mathsf{Y}} - i\mathcal{A}_0(\mathsf{Y}))\Psi(\mathsf{Y}).\end{aligned}$$

It follows

$$Q^{\alpha,\mathcal{A}}\Psi^{\mathcal{A}}(\mathsf{X}) = \mathcal{B}\,\sqrt{\mathcal{B}}\,\exp(i\,\mathcal{G}(\mathsf{X}))\;Q^{\alpha}\Psi(\mathsf{Y}).$$

$\square$

Remark 3.9. Note that $\overline{(\nabla - i\mathcal{A})^2\Psi} = (\nabla + i\mathcal{A})^2\overline{\Psi}$ for any real-valued smooth magnetic potential $\mathcal{A}$ and any $\Psi \in \mathcal{D}(Q^{\alpha,\mathcal{A}})$. Thus, Proposition 3.8 is still valid when the magnetic field $\mathcal{B}$ is negative and $\Psi_k^{\alpha,\mathcal{A}}$ is given on G^α by

$$\Psi_k^{\alpha,\mathcal{A}}(\mathsf{X}) = \sqrt{|\mathcal{B}|}\,\exp(i\,\mathcal{G}(\mathsf{X}))\;\overline{\Psi_k^{\alpha}}(\sqrt{|\mathcal{B}|}\,\mathsf{X}).$$

4 Quasi-modes for the Schrödinger operator with constant magnetic field in a polygonal domain

Before considering in a further step a more general situation (Sections 6-7), we suppose that our domain Ω is a convex bounded polygon with straight edges, and that the magnetic potential is equal to $\mathcal{A}_0(x) = \frac{1}{2}(-x_2, x_1)$. We are interested in the behavior of the lowest eigenvalues of the Neumann realization P_h on Ω, for the Schrödinger operator with magnetic potential $\mathcal{A}_0$ and semi-classical parameter $h > 0$.

The associated sesquilinear form p_h is defined on $H^1(\Omega)$ by

$$p_h(u,v) = \int_\Omega (h\nabla - i\mathcal{A}_0)u(x)\cdot\overline{(h\nabla - i\mathcal{A}_0)v(x)}\,dx. \tag{27}$$

The operator $P_h = -(h\nabla - i\mathcal{A}_0)^2$ is well defined on its domain $\mathcal{D}(P_h)$, with

$$\mathcal{D}(P_h) = \big\{u \in H^2(\Omega), \quad \nu\cdot(h\nabla - i\mathcal{A}_0)u\big|_{\partial\Omega} = 0\big\}. \tag{28}$$

In this section, we introduce suitable corner quasi-modes which will allow to construct limit spectral problems for P_h.

4.1 Definition of corner quasi-modes

Let Σ be the set of the vertices s of Ω, and α_{s} be the opening of Ω at $\mathsf{s} \in \Sigma$. The spectrum of P_h is in close relation with the spectra of the model operators $Q^{\alpha_{\mathsf{s}}}$, as defined in (4), for s describing the set of corners Σ.

As a first step in the explanation of this relation, we introduce, for each vertex s, the infinite plane sector $\breve{G}_{\mathsf{s}}$ which coincides with Ω near the vertex s: For $d > 0$ small enough, we have

$$\Omega \cap B(\mathsf{s}, d) = \breve{G}_{\mathsf{s}} \cap B(\mathsf{s}, d).$$

There exists a rotation $\mathcal{R}_{\mathsf{s}}$ such that

$$\{\mathsf{X} = \mathcal{R}_{\mathsf{s}}(x - \mathsf{s}), \ \ x \in \breve{G}_{\mathsf{s}}\} = G^{\alpha_{\mathsf{s}}}.$$

As a consequence of Proposition 3.8, we obtain:

Lemma 4.1. *For all integer k, $1 \le k \le K_{\alpha_{\mathsf{s}}}$, the function $\breve{\psi}_{h,\mathsf{s},k}$ defined by*

$$\breve{\psi}_{h,\mathsf{s},k}(x) \ = \ \frac{1}{\sqrt{h}} \exp\left(\frac{i}{2h}\, x \wedge \mathsf{s}\right) \ \Psi_k^{\alpha_{\mathsf{s}}}\left(\frac{\mathcal{R}_{\mathsf{s}}(x - \mathsf{s})}{\sqrt{h}}\right) \quad \textit{on} \quad \breve{G}_{\mathsf{s}}, \tag{29}$$

is a normalized eigenvector for the operator $-(h\nabla - i\mathcal{A}_0)^2$ with Neumann boundary conditions on $\breve{G}_{\mathsf{s}}$, associated with the eigenvalue $h\mu_k(\alpha_{\mathsf{s}})$.

Thus we construct quasi-modes for P_h from the eigenpairs $(\mu_k(\alpha_{\mathsf{s}}), \Psi_k^{\alpha_{\mathsf{s}}})$ of $Q^{\alpha_{\mathsf{s}}}$ for each corner s of Ω and each $k \le K_{\alpha_{\mathsf{s}}}$ via translation, rotation and cut-off according to:

Notation 4.2. • Let $\mathsf{s} \in \Sigma$ and ρ_{s} be the distance to other vertices:

$$\rho_{\mathsf{s}} = \operatorname{dist}(\mathsf{s}, \Sigma \setminus \{\mathsf{s}\}).$$

Let $\rho' \in (0, \rho_{\mathsf{s}})$ and χ_{s} be a radial smooth cut-off function with support in $B(\mathsf{s}, \rho_{\mathsf{s}})$, equal to 1 in $B(\mathsf{s}, \rho')$ and such that $0 \le \chi_{\mathsf{s}} \le 1$.
• Let $k \le K_{\alpha_{\mathsf{s}}}$. Applying the cut-off χ_{s} to the function $\breve{\psi}_{h,\mathsf{s},k}$ in (29) we define

$$\psi_{h,\mathsf{s},k}(x) = \chi_{\mathsf{s}}(x)\, \breve{\psi}_{h,\mathsf{s},k}(x) \quad \text{on} \quad \Omega. \tag{30}$$

4.2 Properties of quasi-modes

We gather in the following lemma the main properties of the functions $\psi_{h,\mathsf{s},k}$.

Lemma 4.3. *For any $\varepsilon > 0$, there exists C_ε such that* (31)-(33) *hold.*
(i) The L^2 norm of $\psi_{h,\mathsf{s},k}$ is nearly 1:

$$1 - C_\varepsilon \exp\left(-\frac{2}{\sqrt{h}}\left(\rho'\sqrt{\Theta_0 - \mu_k(\alpha_{\mathsf{s}})} \ - \ \varepsilon\right)\right) \le ||\psi_{h,\mathsf{s},k}||^2_{L^2(\Omega)} \le 1. \tag{31}$$

(ii) The Rayleigh quotient of $\psi_{h,\mathsf{s},k}$ is nearly $h\mu_k(\alpha_{\mathsf{s}})$:

$$\left|\frac{p_h(\psi_{h,\mathsf{s},k}, \psi_{h,\mathsf{s},k})}{||\psi_{h,\mathsf{s},k}||^2_{L^2(\Omega)}} - h\mu_k(\alpha_{\mathsf{s}})\right| \le C_\varepsilon \exp\left(-\frac{2}{\sqrt{h}}\left(\rho'\sqrt{\Theta_0 - \mu_k(\alpha_{\mathsf{s}})} - \varepsilon\right)\right). \tag{32}$$

(iii) The pair $(h\mu_k(\alpha_{\mathsf{s}}), \psi_{h,\mathsf{s},k})$ is an approximate eigenpair of P_h:

$$\frac{||P_h\psi_{h,\mathsf{s},k} - h\mu_k(\alpha_{\mathsf{s}})\psi_{h,\mathsf{s},k}||_{L^2(\Omega)}}{||\psi_{h,\mathsf{s},k}||_{L^2(\Omega)}} \le C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left(\rho'\sqrt{\Theta_0 - \mu_k(\alpha_{\mathsf{s}})} - \varepsilon\right)\right). \tag{33}$$

Proof. *(i)* Since, by construction, $||\breve{\psi}_{h,\mathsf{s},k}||_{L^2(\breve{G}_\mathsf{s})} = 1$ and thanks to the decay properties (6) of $\Psi_k^{\alpha_\mathsf{s}}$, we have

$$\begin{aligned}
1 \geq ||\chi_\mathsf{s}\breve{\psi}_{h,\mathsf{s},k}||^2_{L^2(\Omega)} &\geq \int_{\Omega \cap B(\mathsf{s},\rho')} |\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx \\
&= \int_{G^{\alpha_\mathsf{s}} \cap B\left(0, \frac{\rho'}{\sqrt{h}}\right)} |\Psi_k^{\alpha_\mathsf{s}}(\mathsf{X})|^2 \, d\mathsf{X} \\
&= 1 - \int_{G^{\alpha_\mathsf{s}} \setminus B\left(0, \frac{\rho'}{\sqrt{h}}\right)} |\Psi_k^{\alpha_\mathsf{s}}(\mathsf{X})|^2 \, d\mathsf{X} \\
&\geq 1 - C_\varepsilon \exp\left(-\frac{2\rho'}{\sqrt{h}}\left(\sqrt{\Theta_0 - \mu_k(\alpha_\mathsf{s})} - \varepsilon\right)\right).
\end{aligned}$$

(ii) Let us prove the estimate about the quadratic form. We have $\psi_{h,\mathsf{s},k} \in H^1(\Omega)$ and

$$\begin{aligned}
p_h(\psi_{h,\mathsf{s},k}, \psi_{h,\mathsf{s},k}) &= \int_\Omega |(h\nabla - i\mathcal{A}_0)\psi_{h,\mathsf{s},k}|^2 \, dx \\
&= \int_\Omega |\chi_\mathsf{s}(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx + h^2 \int_\Omega |\breve{\psi}_{h,\mathsf{s},k}|^2 |\nabla\chi_\mathsf{s}|^2 \, dx \\
&\quad + 2h \operatorname{Re} \int_\Omega \chi_\mathsf{s}(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k} \cdot \nabla\chi_\mathsf{s} \, \overline{\breve{\psi}_{h,\mathsf{s},k}} \, dx. \qquad (34)
\end{aligned}$$

Due to the properties of χ_s and to the decay estimate (6) again, we have

$$\begin{aligned}
\int_\Omega |\chi_\mathsf{s}(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx &\leq \int_\Omega |(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx \\
&\leq h\mu_k(\alpha_\mathsf{s}) \, ||\breve{\psi}_{h,\mathsf{s},k}||^2_{L^2(\breve{G}_\mathsf{s})} \qquad (35)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\chi_\mathsf{s}(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx &\geq \int_{\Omega \cap B(\mathsf{s},\rho')} |(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx \\
&\geq \int_{\breve{G}_\mathsf{s}} |(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx - \int_{\breve{G}_\mathsf{s} \setminus B(\mathsf{s},\rho')} |(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 \, dx \\
&\geq h\mu_k(\alpha_\mathsf{s}) \, ||\breve{\psi}_{h,\mathsf{s},k}||^2_{L^2(\breve{G}_\mathsf{s})} - C_\varepsilon \exp\left(-\frac{2\rho'}{\sqrt{h}}\left(\sqrt{\Theta_0 - \mu_k(\alpha_\mathsf{s})} - \varepsilon\right)\right). \qquad (36)
\end{aligned}$$

Still using Theorem 3.2, we deduce also the estimate

$$\begin{aligned}
&\left| 2h \operatorname{Re} \int_\Omega \chi_\mathsf{s}(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k} \cdot \nabla\chi_\mathsf{s} \, \overline{\breve{\psi}_{h,\mathsf{s},k}} \, dx + h^2 \int_\Omega |\breve{\psi}_{h,\mathsf{s},k}|^2 |\nabla\chi_\mathsf{s}|^2 \, dx \right| \\
&\qquad \leq Ch \int_{\Omega \setminus B(\mathsf{s},\rho')} \left(|(h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k}|^2 + |\breve{\psi}_{h,\mathsf{s},k}|^2 \right) dx \\
&\qquad \leq C_\varepsilon \exp\left(-\frac{2\rho'}{\sqrt{h}}\left(\sqrt{\Theta_0 - \mu_k(\alpha_\mathsf{s})} - \varepsilon\right)\right). \qquad (37)
\end{aligned}$$

Putting together relation (34) with estimates (35), (36), (37) and using the estimate (31) for $||\psi_{h,\mathsf{s},k}||_{L^2(\Omega)}$, we deduce (32).

(iii) Let us now prove the last estimate. Since χ_{s} is radial, we can check that $\psi_{h,\mathsf{s},k}$ satisfies the Neumann boundary condition and, thus, belongs to $\mathcal{D}(P_h)$. We have

$$P_h(\psi_{h,\mathsf{s},k}) \quad = \quad \chi_{\mathsf{s}} P_h(\breve{\psi}_{h,\mathsf{s},k}) - 2h\nabla\chi_{\mathsf{s}} \cdot (h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k} - h^2\breve{\psi}_{h,\mathsf{s},k}\,\Delta\chi_{\mathsf{s}}.$$

On the support of χ_{s}, we have, thanks to Lemma 4.1,

$$P_h\breve{\psi}_{h,\mathsf{s},k}(x) = h\mu_k(\alpha_{\mathsf{s}})\breve{\psi}_{h,\mathsf{s},k}(x).$$

Therefore $\chi_{\mathsf{s}} P_h(\breve{\psi}_{h,\mathsf{s},k}) = h\mu_k(\alpha_{\mathsf{s}})\psi_{h,\mathsf{s},k}$. The same arguments as above for the proof of (37) lead to the estimate

$$||2h\nabla\chi_{\mathsf{s}} \cdot (h\nabla - i\mathcal{A}_0)\breve{\psi}_{h,\mathsf{s},k} + h^2\breve{\psi}_{h,\mathsf{s},k}\Delta\chi_{\mathsf{s}}||^2_{L^2(\Omega)}$$
$$\leq C_\varepsilon \exp\left(-\frac{2\rho'}{\sqrt{h}}\left(\sqrt{\Theta_0 - \mu_k(\alpha_{\mathsf{s}})} - \varepsilon\right)\right). \quad (38)$$

This ends the proof of Lemma 4.3. ☐

4.3 Partition of unity

We end this section by a useful lemma which will allow to achieve the proof of the spectral asymptotics which can be obtained from the quasi-modes.

Lemma 4.4. *For any $\mathsf{s} \in \Sigma$, let χ_{s} be a real-valued cut-off function supported in $B(\mathsf{s}, \rho_{\mathsf{s}})$. We assume moreover that for any $\mathsf{s} \neq \mathsf{s}'$, $\operatorname{supp}\chi_{\mathsf{s}} \cap \operatorname{supp}\chi'_{\mathsf{s}} = \emptyset$. We define χ_0 on Ω by $\chi_0^2 = 1 - \sum_{\mathsf{s}\in\Sigma}\chi_{\mathsf{s}}^2$. By convention χ_{s} with $\mathsf{s} = 0$ refers to χ_0. Then, for any $\psi \in H^1(\Omega)$,*

$$p_h(\psi,\psi) = \sum_{\mathsf{s}\in\Sigma\cup\{0\}} p_h(\chi_{\mathsf{s}}\psi, \chi_{\mathsf{s}}\psi) - h^2 \sum_{\mathsf{s}\in\Sigma\cup\{0\}} ||\psi\nabla\chi_{\mathsf{s}}||^2_{L^2(\Omega)}.$$

Proof. Let $\mathsf{s} \in \Sigma \cup \{0\}$, then

$$\begin{aligned}
|(h\nabla - i\mathcal{A}_0)(\chi_{\mathsf{s}}\psi)|^2 \quad &= \quad |\chi_{\mathsf{s}}|^2|(h\nabla - i\mathcal{A}_0)\psi|^2 + h^2|\psi|^2|\nabla\chi_{\mathsf{s}}|^2 \\
&\qquad + 2h \operatorname{Re} \chi_{\mathsf{s}}\overline{\psi}\,(h\nabla - i\mathcal{A}_0)\psi \cdot \nabla\chi_{\mathsf{s}} \\
&= \quad |\chi_{\mathsf{s}}|^2|(h\nabla - i\mathcal{A}_0)\psi|^2 + h^2|\psi|^2|\nabla\chi_{\mathsf{s}}|^2 + h^2 \operatorname{Re} \overline{\psi}\nabla\psi \cdot \nabla|\chi_{\mathsf{s}}|^2.
\end{aligned}$$

Let us sum up this relation for $\mathsf{s} \in \Sigma \cup \{0\}$, it follows

$$\begin{aligned}
\sum_{\mathsf{s}\in\Sigma\cup\{0\}} |(h\nabla - i\mathcal{A}_0)(\chi_{\mathsf{s}}\psi)|^2 \quad &= \quad \sum_{\mathsf{s}\in\Sigma\cup\{0\}} |\chi_{\mathsf{s}}|^2|(h\nabla - i\mathcal{A}_0)\psi|^2 + h^2 \sum_{\mathsf{s}\in\Sigma\cup\{0\}} |\psi|^2|\nabla\chi_{\mathsf{s}}|^2 \\
&\qquad + h^2 \sum_{\mathsf{s}\in\Sigma\cup\{0\}} \operatorname{Re} \overline{\psi}\nabla\psi \cdot \nabla|\chi_{\mathsf{s}}|^2.
\end{aligned}$$

Since $\sum_{\mathsf{s}\in\Sigma\cup\{0\}}|\chi_{\mathsf{s}}|^2=1$ on Ω, we notice that on Ω

$$\begin{aligned}\sum_{\mathsf{s}\in\Sigma\cup\{0\}}|\chi_{\mathsf{s}}|^2|(h\nabla-i\mathcal{A}_0)\psi|^2 &= |(h\nabla-i\mathcal{A}_0)\psi|^2,\\ \sum_{\mathsf{s}\in\Sigma\cup\{0\}}\operatorname{Re}\overline{\psi}\nabla\psi\cdot\nabla|\chi_{\mathsf{s}}|^2 &= \operatorname{Re}\overline{\psi}\nabla\psi\cdot\nabla\sum_{\mathsf{s}\in\Sigma\cup\{0\}}|\chi_{\mathsf{s}}|^2=0.\end{aligned}$$

Integrating on Ω ends the proof. □

5 Spectral asymptotics in a polygon (constant magnetic field)

In this section, we prove that, provided that some of the model operators $Q^{\alpha_{\mathsf{s}}}$ have eigenvalues λ below their essential spectrum, a corresponding number of eigenvalues μ_h of P_h are exponentially close to $h\lambda$ as h tends to 0. We also prove the related results for eigenspaces.

5.1 Approximation of eigenvalues of P_h by corner model operators

We first make precise the notations about eigenvalues.

Notation 5.1. • We denote by $\mu_{h,n}$ the $n-$th eigenvalue of P_h counted with multiplicity.
• We denote by λ_n the $n-$th eigenvalue of $\oplus_{\mathsf{s}\in\Sigma}Q^{\alpha_{\mathsf{s}}}$ counted with multiplicity as defined by the min-max principle, and let K_Ω be the largest integer such that $\lambda_{K_\Omega}<\Theta_0$. With Notation 3.4, we have $K_\Omega=\sum_{\mathsf{s}\in\Sigma}K_{\alpha_{\mathsf{s}}}$. We assume that $K_\Omega\geq 1$. For any $n\leq K_\Omega$, we denote by Σ_n the subset of vertices:

$$\Sigma_n=\big\{\mathsf{s}\in\Sigma,\ \ \lambda_n \text{ is an eigenvalue for } Q^{\alpha_{\mathsf{s}}}\big\},$$

and by r_n the distance

$$r_n=r(\lambda_n)=\min_{\mathsf{s}\in\Sigma_n}d(\mathsf{s},\Sigma\setminus\{\mathsf{s}\}).$$

Theorem 5.2. *With Notation* 5.1*, for any* $\varepsilon>0$*, there exists* C_ε *such that*

$$\begin{aligned}\mu_{h,1} &\leq h\lambda_1+C_\varepsilon\exp\left(-\frac{2}{\sqrt{h}}\left(r_1\sqrt{\Theta_0-\lambda_1}-\varepsilon\right)\right),\\ |\mu_{h,n}-h\lambda_n| &\leq C_\varepsilon\exp\left(-\frac{1}{\sqrt{h}}\left(r_n\sqrt{\Theta_0-\lambda_n}-\varepsilon\right)\right),\qquad \forall n\leq K_\Omega.\end{aligned}$$

Proof. *(i)* Estimate (32) of Lemma 4.3 applied with $\mu_k(\alpha_{\mathsf{s}})=\lambda_1$ and $\rho'=r_1-\varepsilon'$ and the min-max principle (recalled in Definition 3.1) lead to

$$\mu_{h,1}\ \leq\ h\lambda_1+C_\varepsilon\exp\left(-\frac{2}{\sqrt{h}}(r_1-\varepsilon')\left(\sqrt{\Theta_0-\lambda_1}-\varepsilon\right)\right).$$

(ii) Let $n \leq K_\Omega$ and $\mathsf{s} \in \Sigma_n$. Let Ψ^{α_s} be a normalized eigenvector for Q^{α_s} associated with λ_n and let $\breve{\psi}_{h,\mathsf{s}}$ be the function deduced from Ψ^{α_s} by (29). Then $\breve{\psi}_{h,\mathsf{s}}$ is a normalized eigenfunction of $-(h\nabla - i\mathcal{A}_0)^2$ on $\breve{G}_\mathsf{s}$ associated with the eigenvalue $h\lambda_n$. Let $\varepsilon > 0$. Let $\chi_\mathsf{s} \in \mathcal{C}_0^\infty(\Omega, [0,1])$ be a smooth cut-off function as in (30), with $\rho' < r_n - \frac{\varepsilon}{2}$, and define $\psi_{h,\mathsf{s}} = \chi_\mathsf{s}\breve{\psi}_{h,\mathsf{s}}$ as in (30).

We deduce from estimate (33) that there exists $C_\varepsilon > 0$ such that

$$\frac{||P_h(\psi_{h,\mathsf{s}}) - h\lambda_n\psi_{h,\mathsf{s}}||_{L^2(\Omega)}}{||\psi_{h,\mathsf{s}}||_{L^2(\Omega)}} \leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left(r_n\sqrt{\Theta_0 - \lambda_n} - \varepsilon\right)\right). \tag{39}$$

Due to the spectral theorem (cf [25, Chap. VII]), it follows

$$d(\sigma(P_h), h\lambda_n) \leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left(r_n\sqrt{\Theta_0 - \lambda_n} - \varepsilon\right)\right). \tag{40}$$

(iii) Let us prove a lower bound for the eigenvalues of P_h using ideas of [9, 27]. Let $n \leq K_\Omega + 1$ be such that $\lambda_{n-1} \neq \lambda_n$. With the cut-off functions χ_s already introduced for $\mathsf{s} \in \Sigma$, let us define χ_0 on Ω by $\chi_0^2 = 1 - \sum_{\mathsf{s}\in\Sigma}\chi_\mathsf{s}^2$. Due to Lemma 4.4, we know that for any $u \in H^1(\Omega)$,

$$p_h(u,u) = \sum_{\mathsf{s}\in\Sigma\cup\{0\}} p_h(\chi_\mathsf{s}u, \chi_\mathsf{s}u) - h^2 \sum_{\mathsf{s}\in\Sigma\cup\{0\}} ||u\nabla\chi_\mathsf{s}||^2_{L^2(\Omega)}. \tag{41}$$

Since $\operatorname{supp}\chi_0 \cap \Sigma = \emptyset$, we can apply the result of [12] for smooth domains: there exists $c > 0$ so that

$$p_h(\chi_0 u, \chi_0 u) \geq (h\Theta_0 - ch^2)||\chi_0 u||^2_{L^2(\Omega)}. \tag{42}$$

For any $\mathsf{s} \in \Sigma$, let T^s be the restriction of Q^{α_s} to the space spanned by the eigenfunctions $\Psi_k^{\alpha_\mathsf{s}}$ corresponding to eigenvalues $\mu_k(\alpha_\mathsf{s}) \leq \lambda_{n-1}$. We denote by $\mathcal{R}^*_{h,\mathsf{s}}$ the application

$$\begin{array}{rcl} \mathcal{R}^*_{h,\mathsf{s}}\mathcal{D}(P_h) & \to & \mathcal{D}(Q^{\alpha_\mathsf{s}}) \\ \breve{u}_{h,\mathsf{s}} & \mapsto & u \quad \text{such that } \breve{u}_{h,\mathsf{s}}(x) = \dfrac{1}{\sqrt{h}}\exp\left(\dfrac{i}{2h}\,x\wedge\mathsf{s}\right)\,u\left(\dfrac{\mathcal{R}_\mathsf{s}(x-\mathsf{s})}{\sqrt{h}}\right). \end{array}$$

Then we have

$$\begin{aligned} p_h(\chi_\mathsf{s}u, \chi_\mathsf{s}u) &= h\, q^{\alpha_\mathsf{s}}\big(\mathcal{R}^*_{h,\mathsf{s}}(\chi_\mathsf{s}u), \mathcal{R}^*_{h,\mathsf{s}}(\chi_\mathsf{s}u)\big) \\ &\geq h\,\lambda_n||\mathcal{R}^*_{h,\mathsf{s}}(\chi_\mathsf{s}u)||^2_{L^2(G^{\alpha_\mathsf{s}})} - h\left\langle\mathcal{R}^*_{h,\mathsf{s}}(\chi_\mathsf{s}u), T^\mathsf{s}\mathcal{R}^*_{h,\mathsf{s}}(\chi_\mathsf{s}u)\right\rangle. \end{aligned} \tag{43}$$

Let $T_{h,\mathsf{s}}$ be the restriction of $-(h\nabla - i\mathcal{A}_0)^2$ on $\breve{G}_\mathsf{s}$ to the space spanned by the eigenfunctions $\breve{\psi}_{h,\mathsf{s},k}$, see (29), corresponding to eigenvalues $\mu_k(\alpha_\mathsf{s}) \leq \lambda_{n-1}$. We set

$$T = \sum_{\mathsf{s}\in\Sigma}\chi_\mathsf{s}T_{h,\mathsf{s}}\chi_\mathsf{s}.$$

Combining (41)-(43) we obtain, since $\lambda_n \leq \Theta_0$,

$$p_h(u,u) \geq h\lambda_n ||u||^2_{L^2(\Omega)} - \langle u, Tu\rangle - ch^2 ||u||^2_{L^2(\Omega)}.$$

Since for any $\mathsf{s} \in \Sigma$,

$$\text{rank}(\chi_\mathsf{s} T_{h,\mathsf{s}} \chi_\mathsf{s}) \leq \text{rank}(T^\mathsf{s}) = \text{card}\big\{\text{eigenvalues of } Q^{\alpha_\mathsf{s}} \text{ less than } \lambda_{n-1}\big\},$$

we deduce, thanks to the assumption $\lambda_{n-1} < \lambda_n$, that the rank of T is not greater than $n-1$.

To obtain a lower bound for $\mu_{h,n}$, we use the max-min principle. Let $u_1, \dots, u_{n-1}$ belong to the orthogonal space of $\ker(T)$. Then for all ψ in $\{u_1, \dots, u_{n-1}\}^\perp$, ψ belongs to $\ker(T)$ and we have

$$\mu_{h,n} \geq \frac{\langle \psi, P_h \psi\rangle}{||\psi||^2} \geq h\lambda_n - ch^2. \tag{44}$$

(iv) Now we reach the conclusion: According to (40), we know that there exists $\mu_h \in \sigma(P_h)$ such that

$$|\mu_h - h\lambda_n| \leq C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}(r_n\sqrt{\Theta_0 - \lambda_n} - \varepsilon)\right).$$

Using (44), we obtain that μ_h belongs to the set $\{\mu_{h,k},\ \lambda_k = \lambda_n\}$, which ends the proof of the theorem. $\square$

Remark 5.3. In particular the lower bound (44) is valid for $n = K_\Omega + 1$: In this case, $\lambda_{K_\Omega+1} = \Theta_0$. Thus (44) yields

$$\mu_{K_\Omega+1,h} \geq h\Theta_0 - ch^2.$$

5.2 Eigenspaces

It results from the previous theorem that, according to repetitions of the same values in $\lambda \in \{\lambda_1, \dots, \lambda_{K_\Omega}\}$, the corresponding eigenvalues $\mu_{h,n}$ are gathered into clusters, because they are exponentially close to the same value $h\lambda$. We are going to prove that the corresponding eigenvectors are exponentially close to linear combinations of quasi-modes. Let us first introduce the definition of distance between subspaces E and F of a Hilbert space. The (a priori) non-symmetric distance $d(E,F)$ is defined as

$$d(E,F) = ||\Pi_E - \Pi_F\Pi_E||_\mathcal{H},$$

where Π_E and Π_F denote the orthogonal projections on E and F respectively. If both E and F are finite dimensional, and if they have the same dimension, then $d(E,F) = d(F,E)$.

To prove that eigenvectors of P_h are close to linear combinations of quasi-modes, we use the following refinement of [29] more adapted to clustered eigenvalues:

Theorem 5.4 ([15, Prop. 4.1.1] [19]). *Let A be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Let $\psi_1, \dots, \psi_N$ be N linearly independent vectors in $\mathcal{D}(A)$ and $\mu_1, \dots, \mu_N$ be N real numbers such that*

$$A\psi_j = \mu_j \psi_j + r_j \quad \textit{with} \quad ||r_j||_{\mathcal{H}} \le \eta. \tag{45}$$

Let $I \subset \mathbb{R}$ be a compact interval containing $\mu_1, \dots, \mu_N$. We assume that there exists $a > 0$ such that $\sigma(A) \cap (I + B(0, 2a) \setminus I) = \emptyset$. Then, if E is the space spanned by $\psi_1, \dots, \psi_N$ and if F is the spectral space associated with $\sigma(A) \cap I$, we have

$$d(E, F) \le \frac{\eta\sqrt{N}}{a\sqrt{\kappa_S^{\min}}}, \tag{46}$$

where $\kappa_S^{\min}$ is the smallest eigenvalue of the Gram matrix $S = \Big(\langle \psi_j, \psi_k \rangle_{\mathcal{H}}\Big)$.

Let us now introduce some notation for the cluster of eigenspaces and quasi-modes.

Notation 5.5. • Using Notation 5.1, we denote by $\{\Lambda_1 < \cdots < \Lambda_M\}$ the set of distinct values in $\{\lambda_1, \dots, \lambda_{K_\Omega}\}$. For any $m \le M$, we define the distance

$$R_m = r(\Lambda_m).$$

- For any $n \le K_\Omega$, we denote by $(\mu_{h,n}, u_{h,n})$ the n-th eigenpair of P_h.
- For any $m \le M$, we define the m-th cluster of eigenspaces of P_h by

$$F_{h,m} = \text{span}\big\{u_{h,n} \ \text{ for any } \ n \ \text{ such that } \ \lambda_n = \Lambda_m\big\},$$

and the corresponding cluster of quasi-modes, cf. (29)-(30),

$$E_{h,m} = \text{span}\big\{\psi_{h,\mathsf{s},k} = \chi_{\mathsf{s}} \breve{\psi}_{h,\mathsf{s},k} \ \text{ for any } \ \mathsf{s} \in \Sigma, k \ge 1 \ \text{ such that } \ \mu_k(\alpha_{\mathsf{s}}) = \Lambda_m\big\}.$$

A positive real number δ is attached to these spaces of quasi-modes: δ is such that for all $\mathsf{s} \in \Sigma$, the cut-off function χ_{s} is equal to 1 on $B(\mathsf{s}, R_m - \delta)$.

Theorem 5.6. *With Notation 5.1 and 5.5, for any $\varepsilon > 0$, there exists C_ε such that for any $m \le M$,*

$$d(E_{h,m}\,, F_{h,m}) \le C_\varepsilon \exp\left(-\frac{1}{\sqrt{h}}\left((R_m - \delta)\sqrt{\Theta_0 - \Lambda_m} - \varepsilon\right)\right).$$

Proof. For any $m \le M$, we define

$$\Sigma_m^* = \big\{(\mathsf{s}, k) \in \Sigma \times \{1, \cdots, K_\Omega\}, \quad \mu_k(\alpha_{\mathsf{s}}) = \Lambda_m\big\}.$$

(i) If the set Σ_m^* is reduced to one element, then Theorem 5.6 comes from Lemma 4.3.

(ii) We assume that Σ_m^* is not reduced to one element. We denote by $\kappa_m^{\min}$ the smallest eigenvalue of the matrix

$$\Big(\langle\psi_{h,\mathsf{s},k},\ \psi_{h,\mathsf{s}',k'}\rangle\Big)_{((\mathsf{s},k),(\mathsf{s}',k'))\in\Sigma_m^*\times\Sigma_m^*}.$$

Let $(\mathsf{s},k)\neq(\mathsf{s}',k')\in\Sigma_m^*$. If $\mathsf{s}=\mathsf{s}'$, $\breve\psi_{h,\mathsf{s},k}$ and $\breve\psi_{h,\mathsf{s},k'}$ are orthogonal, and thus

$$\int_\Omega \psi_{h,\mathsf{s},k}\,\overline{\psi}_{h,\mathsf{s},k'}\,dx \quad=\quad \int_\Omega (|\chi_\mathsf{s}|^2-1)\,\breve\psi_{h,\mathsf{s},k}\,\overline{\breve\psi}_{h,\mathsf{s},k'}\,dx.$$

Then, by Cauchy-Schwarz inequality and decay of eigenfunctions,

$$\begin{aligned}\left|\int_\Omega \psi_{h,\mathsf{s},k}\,\overline{\psi}_{h,\mathsf{s},k'}\,dx\right| &\le \int_{\Omega\setminus B(\mathsf{s},R_m-\delta)}\left|\breve\psi_{h,\mathsf{s},k}\right|\,\left|\breve\psi_{h,\mathsf{s},k'}\right|\,dx\\ &\le C_\varepsilon\exp\left(-\frac{1}{\sqrt h}\Big(2(R_m-\delta)\sqrt{\Theta_0-\Lambda_m}-\varepsilon\Big)\right).\end{aligned}$$

If $\mathsf{s}\neq\mathsf{s}'$, since eigenfunction $\breve\psi_{h,\mathsf{s},k}$ and $\breve\psi_{h,\mathsf{s}',k'}$ are localized near distinct corner, we have

$$\begin{aligned}\left|\int_\Omega \psi_{h,\mathsf{s},k}\,\overline{\psi}_{h,\mathsf{s}',k'}\,dx\right| &\le \int_\Omega\left|\chi_\mathsf{s}\breve\psi_{h,\mathsf{s},k}\right|\,\left|\chi_{\mathsf{s}'}\breve\psi_{h,\mathsf{s}',k'}\right|\,dx\\ &\le \int_{\Omega\setminus(B(\mathsf{s},R_m-\delta)\,\cup\,B(\mathsf{s}',R_m-\delta))}\left|\breve\psi_{h,\mathsf{s},k}\right|\,\left|\breve\psi_{h,\mathsf{s},k'}\right|\,dx\\ &\le C_\varepsilon\exp\left(-\frac{1}{\sqrt h}\Big(2(R_m-\delta)\sqrt{\Theta_0-\Lambda_m}-\varepsilon\Big)\right).\end{aligned}$$

Using also (31), we deduce that

$$|\kappa_m^{\min}-1|\le C_\varepsilon\exp\left(-\frac{1}{\sqrt h}\Big(2(R_m-\delta)\sqrt{\Theta_0-\Lambda_m}-\varepsilon\Big)\right).$$

Let us now apply Theorem 5.4 with $A=P_h$. Relation (33) in Lemma 4.3 gives (45) with $\eta=C_\varepsilon\exp(-((R_m-\delta)\sqrt{\Theta_0-\Lambda_m}-\varepsilon)/\sqrt h)$. Let us define $I=[h\Lambda_m-\eta,h\Lambda_m+\eta]$. According to Theorem 5.2, there exists $C>0$ such that for h small enough,

$$\sigma(P_h)\cap\big(I+B(0,2Ch)\setminus I\big)=\emptyset.$$

For example, we choose $C=\min\left\{\frac{\Lambda_m-\Lambda_{m-1}}{4},\frac{\Lambda_{m+1}-\Lambda_m}{4}\right\}$ with the convention $\Lambda_0=0$. Assumptions of Theorem 5.4 are filled and this ends the proof of Theorem 5.6. □

Remark 5.7. 1. Theorem 5.6 shows that any eigenfunction of P_h associated with $\mu_{h,n}$ is exponentially close to a linear combination of the quasi-modes corresponding to $\Lambda_m=\lambda_n$ for the model operators. This result is particularly interesting when the polygon Ω has several angles with the same opening. When the polygon presents symmetries, these linear combinations are non trivial, as exhibited by numerical experiments on a square, see [6].

2. Using Proposition 3.8, Theorems 5.2 and 5.6 can easily be generalized to the Schrödinger operator with constant magnetic field equal to $\mathcal{B}$. The eigenvalues are multiplied by $\mathcal{B}$ and we use Proposition 3.8 to construct adapted quasi-modes.

3. Theorems 5.2 and 5.6 are still valid for a non convex polygon, even if the domain of the operator P_h is not contained in $H^2(\Omega)$ any more (a finite number of singular functions have to be added to $H^2(\Omega)$ if non convex angles are present). Moreover, we do not expect the first eigenfunctions of P_h to be localized in a non convex corner, since it is reasonable to conjecture that the bottom of the spectrum of Q^α is equal to Θ_0 for openings α between π and 2π (cf. Remark 3.3).

6 Quasi-modes for the Schrödinger operator in a curvilinear polygon

Let Ω be a bounded curvilinear polygon with a piecewise smooth boundary. As previously, we denote by Σ the set of the vertices s of Ω, and by α_{s} the opening of Ω at s. For the sake of simplicity (cf. Remark 5.7), we assume that $\alpha_{\mathsf{s}} \in (0,\pi)$ for any $\mathsf{s} \in \Sigma$.

Let $\mathcal{B}$ be a smooth positive magnetic field and let $\mathcal{A}$ be a potential associated with $\mathcal{B}$, i.e. $\mathcal{B} = \operatorname{curl}\mathcal{A}$ on $\overline{\Omega}$. As in Sections 4-5, we are interested in the behavior of the eigenpairs of the Neumann realization P_h on Ω, for the Schrödinger operator $-(h\nabla - i\mathcal{A})^2$ as $h \to 0$.

The associated sesquilinear form p_h is defined on $H^1(\Omega)$ by

$$p_h(u,v) = \int_\Omega (h\nabla - i\mathcal{A})u(x) \cdot \overline{(h\nabla - i\mathcal{A})v(x)}\, dx. \tag{47}$$

The operator $P_h = -(h\nabla - i\mathcal{A})^2$ is defined on its domain $\mathcal{D}(P_h)$, with

$$\mathcal{D}(P_h) = \big\{u \in H^2(\Omega), \quad \nu \cdot (h\nabla - i\mathcal{A})u\big|_{\partial\Omega} = 0\big\}. \tag{48}$$

Now, the values of the magnetic field $\mathcal{B}(\mathsf{s})$ at corners s play a key role in the eigenvalue asymptotics. We introduce:

Notation 6.1. • We denote by $\mu_{h,n}$ the n-th eigenvalue of P_h counted with multiplicity.
• We define infimum numbers for the magnetic field by

$$b = \inf_{x\in\overline{\Omega}} \mathcal{B}(x) \quad \text{and} \quad b' = \inf_{x\in\partial\Omega} \mathcal{B}(x).$$

• We denote by λ_n the n-th eigenvalue of the model operator

$$\bigoplus_{\mathsf{s}\in\Sigma} \mathcal{B}(\mathsf{s})\, Q^{\alpha_{\mathsf{s}}},$$

counted with multiplicity as defined by the min-max principle, and let $K_{\Omega,\mathcal{B}}$ be the largest integer K such that

$$\lambda_K < \min(\Theta_0 b', b).$$

We assume that $K_{\Omega,\mathcal{B}} \geq 1$.

For simplicity of the quasi-modes construction, we make the following assumption.

Assumption 6.2. For any $\mathsf{s} \in \Sigma$ and $k \leq K_{\alpha_\mathsf{s}}$ such that $\mathcal{B}(\mathsf{s})\mu_k(\alpha_\mathsf{s}) < \min(\Theta_0 b', b)$, we assume that $\mu_k(\alpha_\mathsf{s})$ is a simple eigenvalue of Q^{α_s}.

The case of the curvilinear polygon is quite different from the polygonal case since a series in power of $\sqrt{h}$ appears now. At each corner $\mathsf{s} \in \Sigma$, we perform the construction of asymptotic quasi-modes in several steps: Section 6.1 is devoted to a change of variables which maps a corner neighborhood to a plane sector. We use a gauge transform in Section 6.2 to reduce to a magnetic field equal to 1 at the vertex. In Section 6.3, we perform a scaling together with a Taylor expansion, and introduce a formal series expression of the eigenmode problem. We solve the formal series equations in Section 6.4 and define subsequently our quasi-modes in Section 6.5.

6.1 Change of variables

Let $\mathsf{s} \in \Sigma$. We consider the change of variables $x \mapsto \hat{x} = \mathcal{R}_\mathsf{s}(x - \mathsf{s})$ which sends s into 0 and $\Omega \cap B(\mathsf{s}, \rho_\mathsf{s})$ onto $G^{\alpha_\mathsf{s}} \cap U_\mathsf{s}$. Here U_s is a neighborhood of 0. This change of variables gives a new magnetic potential $\widehat{\mathcal{A}}$ satisfying

$$\mathcal{A}_1 dx_1 + \mathcal{A}_2 dx_2 = \widehat{\mathcal{A}}_1 d\hat{x}_1 + \widehat{\mathcal{A}}_2 d\hat{x}_2,$$

where $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ and $\widehat{\mathcal{A}} = (\widehat{\mathcal{A}}_1, \widehat{\mathcal{A}}_2)$. This leads to define the new operator

$$\widehat{P}_{h,\mathsf{s}} = -\sum_{j,k=1}^{2} (h\partial_{\hat{x}_j} - i\widehat{\mathcal{A}}_j)\ \sqrt{\det \mathcal{T}_\mathsf{s}}\ \mathcal{T}_{\mathsf{s},jk}\ (h\partial_{\hat{x}_k} - i\widehat{\mathcal{A}}_k),$$

where $\mathcal{T}_\mathsf{s} = d\mathcal{R}_\mathsf{s}^{-1}\ (d\mathcal{R}_\mathsf{s}^{-1})^t$ and $\mathcal{T}_{\mathsf{s},jk}$ are the coefficients of the matrix $\mathcal{T}_\mathsf{s}$. The boundary operator $T_h = \nu \cdot (h\nabla - i\mathcal{A})$ is sent onto

$$\widehat{T}_{h,\mathsf{s}} = \sum_{j,k=1}^{2} \nu_j\ \mathcal{T}_{\mathsf{s},jk}(h\partial_{\hat{x}_k} - i\widehat{\mathcal{A}}_k),$$

where (ν_1, ν_2) is the unit outer normal to the boundary of G^{α_s}.

We observe that

$$\sqrt{\det \mathcal{T}_\mathsf{s}(0)}\ \mathcal{T}_{\mathsf{s},jk}(0) = \delta_{j,k}, \quad \widehat{\mathcal{A}}(0) = \mathcal{A}(\mathsf{s}) \quad \text{and} \quad \mathcal{B}(\mathsf{s}) = \frac{\partial \widehat{\mathcal{A}}_2}{\partial \hat{x}_1}(0) - \frac{\partial \widehat{\mathcal{A}}_1}{\partial \hat{x}_2}(0).$$

For $u \in H^2(\Omega \cap B(\mathsf{s}, \rho_\mathsf{s}))$, let $\widehat{u}$ be defined on $G^{\alpha_\mathsf{s}} \cap U_\mathsf{s}$ by

$$\widehat{u}(\hat{x}) = u(x) = u(\mathcal{R}_\mathsf{s}^{-1}(\hat{x}) + \mathsf{s}).$$

There holds

$$\left\{\begin{array}{rcll} P_h u &=& \mu u & \text{in } \Omega \cap B(\mathsf{s}, \rho_\mathsf{s}) \\ T_h u &=& 0 & \text{on } \partial\Omega \cap B(\mathsf{s}, \rho_\mathsf{s}) \end{array}\right. \quad \Longleftrightarrow \quad \left\{\begin{array}{rcll} \widehat{P}_{h,\mathsf{s}} \widehat{u} &=& \mu \widehat{u} & \text{in } G^{\alpha_\mathsf{s}} \cap U_\mathsf{s} \\ \widehat{T}_{h,\mathsf{s}} \widehat{u} &=& 0 & \text{on } \partial G^{\alpha_\mathsf{s}} \cap U_\mathsf{s}. \end{array}\right.$$

6.2 Gauge transform

To apply Proposition 3.8, we define

$$a_{10} = \mathcal{A}_1(\mathsf{s}) = \widehat{\mathcal{A}}_1(0), \quad a_{11} = \frac{\partial \mathcal{A}_1}{\partial x_1}(\mathsf{s}) = \frac{\partial \widehat{\mathcal{A}}_1}{\partial \hat{x}_1}(0), \quad a_{12} = \frac{\partial \mathcal{A}_1}{\partial x_2}(\mathsf{s}) = \frac{\partial \widehat{\mathcal{A}}_1}{\partial \hat{x}_2}(0),$$
$$a_{20} = \mathcal{A}_2(\mathsf{s}) = \widehat{\mathcal{A}}_2(0), \quad a_{21} = \frac{\partial \mathcal{A}_2}{\partial x_1}(\mathsf{s}) = \frac{\partial \widehat{\mathcal{A}}_2}{\partial \hat{x}_1}(0), \quad a_{22} = \frac{\partial \mathcal{A}_2}{\partial x_2}(\mathsf{s}) = \frac{\partial \widehat{\mathcal{A}}_2}{\partial \hat{x}_2}(0).$$

We also define the gauge function

$$\mathcal{G}_\mathsf{s}(\hat{x}) = \frac{1}{2}\Big(a_{11}\hat{x}_1^2 + a_{22}\hat{x}_2^2 + (a_{12} + a_{21})\hat{x}_1\hat{x}_2\Big) + a_{10}\hat{x}_1 + a_{20}\hat{x}_2.$$

In accordance with Proposition 3.8, we consider the gauge transform which associates with any function $\widehat{u}$ defined on $G^{\alpha_\mathsf{s}} \cap U_\mathsf{s}$, the function $\widetilde{\psi}$ such that

$$\widehat{u}(\hat{x}) = \sqrt{\mathcal{B}(\mathsf{s})}\, \exp\left(\frac{i}{h}\, \mathcal{G}_\mathsf{s}(\hat{x})\right)\ \widetilde{\psi}\left(\sqrt{\mathcal{B}(\mathsf{s})}\, \hat{x}\right). \tag{49}$$

The function $\widetilde{\psi}$ is defined on $G^{\alpha_\mathsf{s}} \cap \widetilde{U}_\mathsf{s}$ now, where $\widetilde{U}_\mathsf{s} = \sqrt{\mathcal{B}(\mathsf{s})} U_\mathsf{s}$. This leads to consider the new coordinates $y = \sqrt{\mathcal{B}(\mathsf{s})}\, \hat{x}$, together with the new magnetic potential

$$\widetilde{\mathcal{A}}(y) = \frac{1}{\sqrt{\mathcal{B}(\mathsf{s})}} \left(\widehat{\mathcal{A}} - \nabla \mathcal{G}_\mathsf{s}\right)\left(\frac{y}{\sqrt{\mathcal{B}(\mathsf{s})}}\right). \tag{50}$$

Thus

$$\widetilde{\mathcal{A}}(0) = 0, \quad \operatorname{curl} \widetilde{\mathcal{A}}(0) = 1 \quad \text{and} \quad \widetilde{\mathcal{A}}(y) = \mathcal{A}_0(y) + \mathcal{O}(y^2) \ \text{ as } \ y \to 0.$$

Introducing the operator $\widetilde{Q}_{h,\mathsf{s}}$ defined on $G^{\alpha_\mathsf{s}} \cap \widetilde{U}_\mathsf{s}$ by

$$\widetilde{Q}_{h,\mathsf{s}} = -\mathcal{B}(\mathsf{s}) \sum_{j,k=1}^{2} \Big(h\partial_{y_j} - i\widetilde{\mathcal{A}}_j(y)\Big)\ g_{\mathsf{s},jk}(y)\ \Big(h\partial_{y_k} - i\widetilde{\mathcal{A}}_k(y)\Big) \tag{51}$$

with $g_{\mathsf{s},jk}(y) = \big(\sqrt{\det \mathcal{T}_\mathsf{s}}\ \mathcal{T}_{\mathsf{s},jk}\big)\big(y\,\mathcal{B}(\mathsf{s})^{-1/2}\big)$, and the corresponding boundary operator $\widetilde{T}_{h,\mathsf{s}}$, we check that

$$\left\{\begin{array}{rcll} \widehat{P}_{h,\mathsf{s}} \widehat{u} &=& \mu \widehat{u} & \text{in } G^{\alpha_\mathsf{s}} \cap U_\mathsf{s} \\ \widehat{T}_{h,\mathsf{s}} \widehat{u} &=& 0 & \text{on } \partial G^{\alpha_\mathsf{s}} \cap U_\mathsf{s} \end{array}\right. \quad \Longleftrightarrow \quad \left\{\begin{array}{rcll} \widetilde{Q}_{h,\mathsf{s}} \widetilde{\psi} &=& \mu \widetilde{\psi} & \text{in } G^{\alpha_\mathsf{s}} \cap \widetilde{U}_\mathsf{s} \\ \widetilde{T}_{h,\mathsf{s}} \widetilde{\psi} &=& 0 & \text{on } \partial G^{\alpha_\mathsf{s}} \cap \widetilde{U}_\mathsf{s}. \end{array}\right.$$

6.3 Scaling and formal series expression

In order to homogenize the powers of h in the leading part of $\widetilde{Q}_{h,\mathsf{s}}$ at 0, we have to perform the scaling

$$\mathsf{Y} = \frac{y}{\sqrt{h}} = \sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\,\hat{x}.$$

Now appear the terms $\widetilde{\mathcal{A}}(\sqrt{h}\,\mathsf{Y})$ and $g_{\mathsf{s},jk}(\sqrt{h}\,\mathsf{Y})$, which are not homogeneous functions any more, in general. We represent them as asymptotic series of homogeneous functions thanks to a Taylor expansion at 0. We denote

$$y^\ell = y_1^{\ell_1} y_2^{\ell_2} \quad \text{for} \quad y = (y_1, y_2) \;\text{ and }\; \ell = (\ell_1, \ell_2) \in \mathbb{N}^2.$$

We write the Taylor expansion of the potential $\widetilde{\mathcal{A}}$ and of the change of variables $g_{\mathsf{s},jk}$: For any $j,k \in \{1,2\}$, there exist sequences $(a_j^\ell)_{\ell \geq 1}$ and $(g_{j,k}^\ell)_{\ell \geq 0}$ satisfying, in the sense of asymptotic series

$$\begin{cases} \widetilde{\mathcal{A}}_j(y) & \simeq & \displaystyle\sum_{|\ell| \geq 1} a_j^\ell\, y^\ell, \\ g_{\mathsf{s},jk}(y) & \simeq & \displaystyle\sum_{|\ell| \geq 0} g_{j,k}^\ell\, y^\ell \quad \text{with} \;\; g_{j,k}^0 = \delta_{j,k}. \end{cases} \tag{52}$$

With the scaling $y = \sqrt{h}\,\mathsf{Y}$, we have

$$\widetilde{\mathcal{A}}_j(\sqrt{h}\,\mathsf{Y}) \simeq \sum_{|\ell| \geq 1} h^{\ell/2}\, a_j^\ell\, \mathsf{Y}^\ell$$

and a similar formula for $g_{\mathsf{s},jk}(\sqrt{h}\,\mathsf{Y})$. Then the operator $\widehat{Q}_{h,\mathsf{s}}$ defined in (51) can be written in the form of a formal series of $\sqrt{h}$ as

$$\widetilde{Q}_{h,\mathsf{s}} \simeq h\mathcal{B}(\mathsf{s})\; Q_{\mathsf{s}}[h] \quad \text{where} \quad Q_{\mathsf{s}}[h] = \sum_{\ell \geq 0} h^{\ell/2} Q_{\mathsf{s}}^\ell \quad \text{with} \quad Q_{\mathsf{s}}^0 = Q^{\alpha_{\mathsf{s}}}.$$

We note that, for any $\ell \geq 1$, Q_{s}^ℓ is a second order operator whose coefficients are polynomial functions. We do the same with the boundary operator:

$$\widetilde{T}_{h,\mathsf{s}} \simeq \sqrt{h\mathcal{B}(\mathsf{s})}\; T_{\mathsf{s}}[h] \quad \text{where} \quad T_{\mathsf{s}}[h] = \sum_{\ell \geq 0} h^{\ell/2} T_{\mathsf{s}}^\ell \quad \text{with} \quad T_{\mathsf{s}}^0 = \nu \cdot (\nabla - i\mathcal{A}_0).$$

Putting all together we obtain:

Lemma 6.3. *Let us suppose that there exists a family of solutions* $\big(\mu_h, \widetilde{\psi}_h\big)_{0<h<h_0}$ *to*

$$\begin{cases} \widetilde{Q}_{h,\mathsf{s}}\, \widetilde{\psi}_h & = & h\mathcal{B}(\mathsf{s})\, \mu_h\, \widetilde{\psi}_h \qquad \textit{in } \; G^{\alpha_{\mathsf{s}}} \cap \widetilde{U}_{\mathsf{s}} \\ \widetilde{T}_{h,\mathsf{s}}\, \widetilde{\psi} & = & 0 \qquad\qquad\qquad\; \textit{on } \; \partial G^{\alpha_{\mathsf{s}}} \cap \widetilde{U}_{\mathsf{s}}, \end{cases}$$

such that μ_h and $\widetilde{\psi}_h$ have power series expansions $\mu[h]$ and $\widetilde{\psi}[h]$ in powers of $\sqrt{h}$. Then there holds the formal series equation:

$$\left\{\begin{array}{rcll} Q_{\mathsf{s}}[h]\,\widetilde{\psi}[h] & = & \mu[h]\,\widetilde{\psi}[h] & \text{in } G^{\alpha_{\mathsf{s}}}, \\ T_{\mathsf{s}}[h]\,\widetilde{\psi}[h] & = & 0 & \text{on } \partial G^{\alpha_{\mathsf{s}}}, \end{array}\right. \tag{53}$$

which means that for all integer $L \geq 0$ there holds

$$\left\{\begin{array}{rcll} \displaystyle\sum_{\ell=0}^{L} Q_{\mathsf{s}}^{L-\ell}\,\widetilde{\psi}^{\ell} & = & \displaystyle\sum_{\ell=0}^{L} \mu^{L-\ell}\,\widetilde{\psi}^{\ell} & \text{in } G^{\alpha_{\mathsf{s}}}, \\ \displaystyle\sum_{\ell=0}^{L} T_{\mathsf{s}}^{L-\ell}\,\widetilde{\psi}^{\ell} & = & 0 & \text{on } \partial G^{\alpha_{\mathsf{s}}}. \end{array}\right. \tag{54}$$

Conversely, our quasi-modes are obtained from particular solutions of (53).

6.4 Solutions of the formal series equation (53)

The first problem in (54) (for $L = 0$) is

$$\left\{\begin{array}{rcll} Q_{\mathsf{s}}^{0}\,\widetilde{\psi}^{0} & = & \mu^{0}\,\widetilde{\psi}^{0} & \text{in } G^{\alpha_{\mathsf{s}}}, \\ T_{\mathsf{s}}^{0}\,\widetilde{\psi}^{0} & = & 0 & \text{on } \partial G^{\alpha_{\mathsf{s}}}. \end{array}\right. \tag{55}$$

The solutions of this problem are the eigenpairs $(\mu_k(\alpha_{\mathsf{s}}), \Psi_k^{\alpha_{\mathsf{s}}})$ of the operator $Q^{\alpha_{\mathsf{s}}}$, which we take as starting values for our construction of solutions of the whole system (54). We fix $k \leq K_{\alpha_{\mathsf{s}}}$ and set

$$\mu_{\mathsf{s},k}^{0} = \mu_k(\alpha_{\mathsf{s}}) \quad \text{and} \quad \Psi_{\mathsf{s},k}^{0} = \Psi_k^{\alpha_{\mathsf{s}}}.$$

Then $(\mu_{\mathsf{s},k}^{0}, \Psi_{\mathsf{s},k}^{0})$ is a particular solution of (55). For simplicity, we assume that $\mu_k(\alpha_{\mathsf{s}})$ is a simple eigenvalue of $Q^{\alpha_{\mathsf{s}}}$ (see Assumption 6.2 and Remark 7.2). We determine successively the terms $\Psi_{\mathsf{s},k}^{\ell}$ and $\mu_{\mathsf{s},k}^{\ell}$ of the formal series

$$\Psi_{\mathsf{s},k}[h] = \sum_{\ell \geq 0} h^{\ell/2} \Psi_{\mathsf{s},k}^{\ell} \quad \text{and} \quad \mu_{\mathsf{s},k}[h] = \sum_{\ell \geq 0} h^{\ell/2} \mu_{\mathsf{s},k}^{\ell}$$

by solving successively each problem in the sequence of problems (54), $L = 1, 2, \ldots$

For $L = 1$, the problem is

$$\left\{\begin{array}{rcl} Q_{\mathsf{s}}^{1}\Psi_{\mathsf{s},k}^{0} + Q_{\mathsf{s}}^{0}\Psi_{\mathsf{s},k}^{1} & = & \mu_{\mathsf{s},k}^{1}\Psi_{\mathsf{s},k}^{0} + \mu_{\mathsf{s},k}^{0}\Psi_{\mathsf{s},k}^{1}, \\ T_{\mathsf{s}}^{1}\Psi_{\mathsf{s},k}^{0} + T_{\mathsf{s}}^{0}\Psi_{\mathsf{s},k}^{1} & = & 0. \end{array}\right.$$

We can rewrite it as

$$\left\{\begin{array}{rcl} (Q_{\mathsf{s}}^{0} - \mu_k(\alpha_{\mathsf{s}}))\Psi_{\mathsf{s},k}^{1} & = & (\mu_{\mathsf{s},k}^{1} - Q_{\mathsf{s}}^{1})\Psi_k^{\alpha_{\mathsf{s}}}, \\ T_{\mathsf{s}}^{0}\Psi_{\mathsf{s},k}^{1} & = & -T_{\mathsf{s}}^{1}\Psi_k^{\alpha_{\mathsf{s}}}. \end{array}\right. \tag{56}$$

This problem has the form of problem (21) considered in Lemma 3.7, with the right hand side

$$\mathcal{L}(v) = \left\langle (\mu^1_{\mathsf{s},k} - Q^1_{\mathsf{s}})\Psi^{\alpha_{\mathsf{s}}}_k, v\right\rangle_{G^{\alpha_{\mathsf{s}}}} + \left\langle T^1_{\mathsf{s}}\Psi^{\alpha_{\mathsf{s}}}_k, v\right\rangle_{\partial G^{\alpha_{\mathsf{s}}}}.$$

We choose $\mu^1_{\mathsf{s},k}$ so that the compatibility condition (19) is satisfied, i.e.

$$\mu^1_{\mathsf{s}} = \left\langle Q^1_{\mathsf{s}}\Psi^{\alpha_{\mathsf{s}}}_k, \Psi^{\alpha_{\mathsf{s}}}_k\right\rangle_{G^{\alpha_{\mathsf{s}}}} - \left\langle T^1_{\mathsf{s}}\Psi^{\alpha_{\mathsf{s}}}_k, \Psi^{\alpha_{\mathsf{s}}}_k\right\rangle_{\partial G^{\alpha_{\mathsf{s}}}}.$$

By decay of $\Psi^{\alpha_{\mathsf{s}}}_k$ and since Q^1_{s} is an operator of order 2, the right hand side $\mathcal{L}$ satisfies the decay condition (20). Thus Lemma 3.7 provides the existence of a solution $\Psi^1_{\mathsf{s},k}$ satisfying (56) and, furthermore, there exists $\delta_1 > 0$ such that $\mathrm{e}^{\delta_1|\mathsf{X}|}\Psi^1_{\mathsf{s},k} \in \mathcal{V}(q^{\alpha_{\mathsf{s}}})$.

Next, the problem for $L = 2$ can be written as

$$\left\{\begin{array}{rcl} (Q^0_{\mathsf{s}} - \mu_k(\alpha_{\mathsf{s}}))\Psi^2_{\mathsf{s},k} &=& (\mu^2_{\mathsf{s},k} - Q^2_{\mathsf{s}})\Psi^{\alpha_{\mathsf{s}}}_k + (\mu^1_{\mathsf{s},k} - Q^1_{\mathsf{s}})\Psi^1_{\mathsf{s},k}, \\ T^0_{\mathsf{s}}\Psi^2_{\mathsf{s},k} &=& -T^1_{\mathsf{s}}\Psi^1_{\mathsf{s},k} - T^2_{\mathsf{s}}\Psi^{\alpha_{\mathsf{s}}}_k. \end{array}\right.$$

We choose $\mu^2_{\mathsf{s},k}$ to satisfy (19). Due to the decay of $\Psi^1_{\mathsf{s},k}$ and $\Psi^{\alpha_{\mathsf{s}}}_k$, (20) is satisfied and we apply Lemma 3.7 to determine $\Psi^2_{\mathsf{s},k}$. We do the same successively and so determine coefficients $\Psi^\ell_{\mathsf{s},k}$ and $\mu^\ell_{\mathsf{s},k}$. Construction is summed up in the following lemma.

Lemma 6.4. *Let $\mathsf{s} \in \Sigma$. We fix $k \leq K_{\alpha_{\mathsf{s}}}$ and set*

$$\mu^0_{\mathsf{s},k} = \mu_k(\alpha_{\mathsf{s}}) \quad \textit{and} \quad \Psi^0_{\mathsf{s},k} = \Psi^{\alpha_{\mathsf{s}}}_k.$$

We assume that $\mu_k(\alpha_{\mathsf{s}})$ is simple for $Q^{\alpha_{\mathsf{s}}}$ (see Assumption 6.2).
For any $\ell \geq 1$, there exist $\mu^\ell_{\mathsf{s},k} \in \mathbb{R}$, $\Psi^\ell_{\mathsf{s},k} \in \mathcal{V}(q^{\alpha_{\mathsf{s}}})$ such that for any $L \geq 0$,

$$\left\{\begin{array}{rcll} \displaystyle\sum_{\ell=0}^{L} Q^{L-\ell}_{\mathsf{s}}\,\Psi^\ell_{\mathsf{s},k} &=& \displaystyle\sum_{\ell=0}^{L} \mu^{L-\ell}_{\mathsf{s},k}\,\Psi^\ell_{\mathsf{s},k} & \textit{in } G^{\alpha_{\mathsf{s}}}, \\ \displaystyle\sum_{\ell=0}^{L} T^{L-\ell}_{\mathsf{s}}\,\Psi^\ell_{\mathsf{s},k} &=& 0 & \textit{on } \partial G^{\alpha_{\mathsf{s}}}. \end{array}\right. \tag{57}$$

Furthermore, there exists $\delta > 0$ such that $\mathrm{e}^{\delta|\mathsf{X}|}\Psi^\ell_{\mathsf{s},k} \in \mathcal{V}(q^{\alpha_{\mathsf{s}}})$ for any $\ell \geq 0$.

6.5 Sequences of quasi-modes for P_h near the corner s

We are now ready to introduce sequences of quasi-modes:

Notation 6.5. • Let $\mathsf{s} \in \Sigma$ and let χ_{s} be a smooth cut-off function with support in $B(\mathsf{s}, \rho_{\mathsf{s}})$, equal to 1 in $B(\mathsf{s}, \rho_{\mathsf{s}}/2)$.

• Let $k \leq K_{\alpha_{\mathsf{s}}}$. For any integer $L \geq 0$, using the numbers $\mu^{\ell}_{\mathsf{s},k}$ and functions $\Psi^{\ell}_{\mathsf{s},k}$ exhibited in Lemma 6.4, we define

$$\mu^{[L]}_{h,\mathsf{s},k} = h\mathcal{B}(\mathsf{s}) \sum_{\ell=0}^{L} h^{\ell/2}\mu^{\ell}_{\mathsf{s},k}, \tag{58}$$

$$\breve{\psi}^{[L]}_{h,\mathsf{s},k}(x) = \sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}} \exp\left(\frac{i}{h}\,\mathcal{G}_{\mathsf{s}}(\mathcal{R}_{\mathsf{s}}(x-\mathsf{s}))\right) \sum_{\ell=0}^{L} h^{\ell/2}\Psi^{\ell}_{\mathsf{s},k}\left(\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\,\mathcal{R}_{\mathsf{s}}(x-\mathsf{s})\right) \tag{59}$$

and, applying the cut-off χ_{s}

$$\psi^{[L]}_{h,\mathsf{s},k}(x) = \chi_{\mathsf{s}}(x)\,\breve{\psi}^{[L]}_{h,\mathsf{s},k}(x) \quad \text{on} \quad \Omega. \tag{60}$$

For each fixed corner s and $k \leq K_{\alpha_{\mathsf{s}}}$, all the functions $\psi^{[L]}_{h,\mathsf{s},k}$ for $L = 1, 2, \dots$ are quasi-modes, more and more accurate as $L \to \infty$, as proved in the following lemma:

Lemma 6.6. *Let $\mathsf{s} \in \Sigma$ and $k \leq K_{\alpha_{\mathsf{s}}}$. For any $L \geq 1$, there exists $C_L > 0$ such that* (61)-(62) *hold.*
(i) The L^2-norm of $\psi^{[L]}_{h,\mathsf{s},k}$ is nearly 1:

$$\forall h \in (0, h_0),\ 1 - C_L\sqrt{h} \leq ||\psi^{[L]}_{h,\mathsf{s},k}||_{L^2(\Omega)} \leq 1 + C_L\sqrt{h}, \tag{61}$$

(ii) The Rayleigh quotient of $\psi^{[L]}_{h,\mathsf{s},k}$ is nearly $\mu^{[L]}_{h,\mathsf{s},k}$:

$$\forall h \in (0, h_0),\ \left|p_h(\psi^{[L]}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s},k}) - \mu^{[L]}_{h,\mathsf{s},k}\right| \leq C_L\ h^{\frac{L+3}{2}}. \tag{62}$$

Proof. *(i)* Thanks to Lemma 6.4, we have

$$1 \geq ||\psi^{[0]}_{h,\mathsf{s},k}||^2_{L^2(\Omega)} \geq ||\breve{\psi}^{[0]}_{h,\mathsf{s},k}||^2_{L^2(\Omega \cap B(\mathsf{s},\rho_{\mathsf{s}}/2))} \geq 1 - C\mathrm{e}^{-\delta_0\rho_{\mathsf{s}}\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}}.$$

To end the proof of (61), we notice that $\psi^{[L]}_{h,\mathsf{s},k} = \psi^{[0]}_{h,\mathsf{s},k} + \sqrt{h}v$, with

$$v(x) = \chi_{\mathsf{s}}(x)\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}} \exp\left(\frac{i}{h}\,\mathcal{G}_{\mathsf{s}}(\mathcal{R}_{\mathsf{s}}(x-\mathsf{s}))\right) \sum_{\ell=1}^{L} h^{(\ell-1)/2}\Psi^{\ell}_{\mathsf{s},k}\left(\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\,\mathcal{R}_{\mathsf{s}}(x-\mathsf{s})\right).$$

Consequently, there exists C_L such that $||v||_{L^2(\Omega)} \leq C_L$.
(ii) Let us prove the quadratic form estimate. We have $\psi^{[L]}_{h,\mathsf{s},k} \in H^1(\Omega)$ and we observe, as in (34), that

$$\begin{aligned} p_h(\psi^{[L]}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s},k}) &= \int_\Omega |\chi_{\mathsf{s}}(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2\,dx + h^2\int_\Omega |\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2|\nabla\chi_{\mathsf{s}}|^2\,dx \\ &\quad + 2h\,\mathrm{Re}\int_\Omega \chi_{\mathsf{s}}(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}\cdot\nabla\chi_{\mathsf{s}}\,\overline{\breve{\psi}}^{[L]}_{h,\mathsf{s},k}\,dx. \end{aligned} \tag{63}$$

Due to the properties of χ_{s} and Lemma 6.4, we have

$$\begin{aligned}\int_\Omega |\chi_{\mathsf{s}}(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2\,dx &\le \int_\Omega |(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2\,dx \\ &\le (\mu^{[L]}_{h,\mathsf{s},k} + C_L h^{\frac{L+3}{2}})\,||\breve{\psi}^{[L]}_{h,\mathsf{s},k}||^2_{L^2(\Omega)} \end{aligned}\tag{64}$$

and there exists $\delta > 0$ such that

$$\begin{aligned}\int_\Omega |\chi_{\mathsf{s}}(h\nabla - i\mathcal{A})\breve{\psi}_{h,\mathsf{s},k}|^2\,dx &\ge \int_{\Omega\cap B(\mathsf{s},\frac{\rho_{\mathsf{s}}}{2})} |(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2\,dx \\ &\ge (\mu^{[L]}_{h,\mathsf{s},k} - C_L h^{\frac{L+3}{2}})\,||\breve{\psi}^{[L]}_{h,\mathsf{s},k}||^2_{L^2(\Omega)} - C_L\exp\left(-\delta\rho_{\mathsf{s}}\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\right). \end{aligned}\tag{65}$$

Still using the decay of $\Psi^{\ell}_{\mathsf{s},k}$, we deduce also the estimate

$$\begin{aligned}&\left| 2h\ \mathrm{Re}\int_\Omega \chi_{\mathsf{s}}(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}\cdot\nabla\chi_{\mathsf{s}}\ \overline{\breve{\psi}}^{[L]}_{h,\mathsf{s},k}\,dx + h^2\int_\Omega |\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2|\nabla\chi_{\mathsf{s}}|^2\,dx\right| \\ &\qquad\le Ch\int_{\Omega\setminus B(\mathsf{s},\frac{\rho_{\mathsf{s}}}{2})}\left(|(h\nabla - i\mathcal{A})\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2 + |\breve{\psi}^{[L]}_{h,\mathsf{s},k}|^2\right)dx \\ &\qquad\le C\exp\left(-\delta\rho_{\mathsf{s}}\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\right). \end{aligned}\tag{66}$$

Putting together relation (63) with estimates (64), (65), (66) and using the estimate (61) for $||\psi^{[L]}_{h,\mathsf{s},k}||_{L^2(\Omega)}$, we deduce (62). □

As in Lemma 4.3, we would like to propose an approximate eigenpair of P_h from $(\mu^{[L]}_{\mathsf{s},k}, \Psi^{[L]}_{\mathsf{s},k})$. But $\Psi^{[L]}_{\mathsf{s},k}$ does not fulfill boundary condition and so we introduce a small corrector by the following way:

Notation 6.7. Let $\mathsf{s} \in \Sigma$ and $k \le K_{\alpha_{\mathsf{s}}}$. With Notation 6.5, for any integer $L \ge 0$, we consider $w^L_{h,\mathsf{s},k}$ the solution of the Neumann problem:

$$(h\nabla - i\mathcal{A})^2 w^L_{h,\mathsf{s},k} = 0 \text{ on } \Omega \quad\text{and}\quad \nu\cdot(h\nabla - i\mathcal{A})(\psi^{[L]}_{h,\mathsf{s},k} + w^L_{h,\mathsf{s},k}) = 0 \text{ on } \partial\Omega.$$

Then, we define

$$\phi^{[L]}_{h,\mathsf{s},k} = \psi^{[L]}_{h,\mathsf{s},k} + w^L_{h,\mathsf{s},k}. \tag{67}$$

In the following lemma, we prove that $w^L_{h,\mathsf{s},k}$ is small.

Lemma 6.8. *With Notation 6.7, there exists C such that*

$$||w^L_{h,\mathsf{s},k}||_{L^2(\Omega)} \le Ch^{\frac{L-1}{2}}.$$

Proof. For any $v \in H^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega (-(h\nabla - i\mathcal{A})^2 w^L_{h,\mathsf{s},k})\,\overline{v}\,dx &= 0\\ &= p_h(w^L_{h,\mathsf{s},k}, v) + \int_{\partial\Omega} \nu\cdot(h\nabla - i\mathcal{A})\psi^{[L]}_{h,\mathsf{s},k}\,\overline{v}\,d\Gamma.\end{aligned}$$

This leads to

$$p_h(w^L_{h,\mathsf{s},k}, w^L_{h,\mathsf{s},k}) \leq ||\nu\cdot(h\nabla - i\mathcal{A})\psi^{[L]}_{h,\mathsf{s},k}||_{H^{-1/2}(\partial\Omega)}\ ||w^L_{h,\mathsf{s},k}||_{H^{1/2}(\partial\Omega)}.$$

Using the lower bound of $\mu_{h,1}$ given in [5], there exists $C > 0$ such that

$$p_h(w^L_{h,\mathsf{s},k}, w^L_{h,\mathsf{s},k}) \geq Ch\ ||w^L_{h,\mathsf{s},k}||^2.$$

Furthermore, since Ω is bounded, we have

$$p_h(w^L_{h,\mathsf{s},k}, w^L_{h,\mathsf{s},k}) \geq \tilde{C}h^2||w^L_{h,\mathsf{s},k}||^2_{H^1(\Omega)}.$$

By construction of $\breve{\psi}^{[L]}_{h,\mathsf{s},k}$, localization of χ_s and Lemma 6.4, there exists $C' > 0$ such that

$$||\nu\cdot(h\nabla - i\mathcal{A})\psi^{[L]}_{h,\mathsf{s},k}||_{H^{-1/2}(\partial\Omega)} \leq C' h^{\frac{L+2}{2}}.$$

Consequently,

$$||w^L_{h,\mathsf{s},k}|| \leq ch^{\frac{L-1}{2}}.$$

□

Using the corrector $w^L_{h,\mathsf{s},k}$, we will now establish that quasi-modes $\phi^{[L]}_{h,\mathsf{s},k}$ approximate more and more accurately an eigenfunction of P_h as $L \to \infty$.

Lemma 6.9. *Let $\mathsf{s}, \mathsf{s}' \in \Sigma$ and $k \leq K_{\alpha_\mathsf{s}}$, $k' \leq K_{\alpha_\mathsf{s}}$. For any $L \geq 2$, there exists $C_L > 0$ such that $\phi^{[L]}_{h,\mathsf{s},k} \in \mathcal{D}(P_h)$ and* (68)-(69) *hold.*
(i) Quasi-orthogonality:

$$|\langle \phi^{[L]}_{h,\mathsf{s},k}, \phi^{[L]}_{h,\mathsf{s}',k'}\rangle - \delta_{\mathsf{s},\mathsf{s}'}\ \delta_{k,k'}| \leq C_L h^{1/2}, \tag{68}$$

with the convention $\delta_{\mathsf{s},\mathsf{s}'} = 1$ if $\mathsf{s} = \mathsf{s}'$ and 0 else.
(ii) The pair $(\mu^{[L]}_{h,\mathsf{s},k}, \phi^{[L]}_{h,\mathsf{s},k})$ is an approximate eigenpair of P_h:

$$\left|\left| P_h \phi^{[L]}_{h,\mathsf{s},k} - \mu^{[L]}_{h,\mathsf{s},k}\ \phi^{[L]}_{h,\mathsf{s},k}\right|\right|_{L^2(\Omega)} \leq C_L\ h^{\frac{L+1}{2}}. \tag{69}$$

Proof. By construction of $\Psi^{\ell}_{\mathsf{s},k}$ (cf. Lemma 6.4), the definition of $\psi^{[L]}_{h,\mathsf{s},k}$ and $w^{L}_{h,\mathsf{s},k}$ in Notation 6.5, we have $\Phi^{[L]}_{h,\mathsf{s},k} \in \mathcal{D}(P_h)$.
(i) Let us prove quasi-orthogonality:

$$\begin{aligned}\langle \phi^{[L]}_{h,\mathsf{s},k}, \phi^{[L]}_{h,\mathsf{s}',k'} \rangle \\ = \langle \psi^{[L]}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s}',k'} \rangle + \langle \psi^{[L]}_{h,\mathsf{s},k}, w^{L}_{h,\mathsf{s}',k'} \rangle + \langle w^{L}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s}',k'} \rangle + \langle w^{L}_{h,\mathsf{s},k}, w^{L}_{h,\mathsf{s}',k'} \rangle. \end{aligned} \tag{70}$$

With Notation 6.5 and Lemma 6.4, we have

$$\left| \langle \psi^{[L]}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s}',k'} \rangle - \delta_{\mathsf{s},\mathsf{s}'} \delta_{k,k'} \right| \leq C_L h^{1/2}.$$

By construction of $w^{L}_{h,\mathsf{s},k}$ and Lemma 6.8, we have also

$$\left| \langle \psi^{[L]}_{h,\mathsf{s},k}, w^{L}_{h,\mathsf{s}',k'} \rangle + \langle w^{L}_{h,\mathsf{s},k}, \psi^{[L]}_{h,\mathsf{s}',k'} \rangle + \langle w^{L}_{h,\mathsf{s},k}, w^{L}_{h,\mathsf{s}',k'} \rangle \right| \leq C_L h^{\frac{L-1}{2}}.$$

(ii) Let us prove the second estimate. We have

$$\begin{aligned} P_h \phi^{[L]}_{h,\mathsf{s},k} - \mu^{[L]}_{h,\mathsf{s},k} \phi^{[L]}_{h,\mathsf{s},k} &= (P_h - \mu^{[L]}_{h,\mathsf{s},k}) \psi^{[L]}_{h,\mathsf{s},k} + P_h w^{L}_{h,\mathsf{s},k} - \mu^{[L]}_{h,\mathsf{s},k} w^{L}_{h,\mathsf{s},k} \\ &= \chi_{\mathsf{s}} (P_h - \mu^{[L]}_{h,\mathsf{s},k}) \breve{\psi}^{[L]}_{h,\mathsf{s},k} \\ &\quad -2h \nabla \chi_{\mathsf{s}} \cdot (h\nabla - i\mathcal{A}) \breve{\psi}^{[L]}_{h,\mathsf{s},k} - h^2 \breve{\psi}^{[L]}_{h,\mathsf{s},k} \Delta \chi_{\mathsf{s}} - \mu^{[L]}_{h,\mathsf{s},k} w^{L}_{h,\mathsf{s},k}, \end{aligned}$$

since $P_h w^{L}_{h,\mathsf{s},k} = 0$. By localization of the support of χ_{s}, we have

$$||\chi_{\mathsf{s}} P_h \breve{\psi}^{[L]}_{h,\mathsf{s},k} - \mu^{[L]}_{h,\mathsf{s},k} \psi^{[L]}_{h,\mathsf{s},k}||_{L^2(\Omega)} \leq C_L h^{\frac{L+3}{2}}.$$

By decay of $\Psi^{\ell}_{\mathsf{s},k}$ given in Lemma 6.4 and localization of the support of $\nabla \chi_{\mathsf{s}}$ and $\Delta \chi_{\mathsf{s}}$, there exists $C_L > 0$ such that

$$||2h \nabla \chi_{\mathsf{s}} \cdot (h\nabla - i\mathcal{A}) \breve{\psi}_{h,\mathsf{s},k} + h^2 \breve{\psi}_{h,\mathsf{s},k} \Delta \chi_{\mathsf{s}}||^2_{L^2(\Omega)} \leq C_L \exp\left(-\delta \rho_{\mathsf{s}} \sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}} \right).$$

By Lemma 6.8, we have

$$\mu^{[L]}_{h,\mathsf{s},k} ||w^{L}_{h,\mathsf{s},k}||_{L^2(\Omega)} \leq C h^{\frac{L+1}{2}}.$$

□

7 Spectral asymptotics in a curvilinear polygon

In this section, we give the asymptotics of the low-lying eigenvalues of P_h using the model operators $Q^{\alpha_{\mathsf{s}}}$. We also prove the related results for eigenspaces.

7.1 Eigenvalue asymptotics

Theorem 7.1. *Let $L \geq 2$. With Notation 6.1 and Assumption 6.2, let $\mathcal{E}^L(h)$ be the set of the $K_{\Omega,\mathcal{B}}$ smallest eigenvalue asymptotics $\mu^{[L]}_{h,\mathsf{s},k}$ as defined in Notation 6.5:*

$$\mathcal{E}^L(h) = \{\mu^{[L]}_{h,\mathsf{s},k}\,,\ \mathsf{s} \in \Sigma, k \leq K_{\alpha_\mathsf{s}} \quad \textit{such that} \quad \mathcal{B}(\mathsf{s})\mu_k(\alpha_\mathsf{s}) < \min(b'\Theta_0, b)\}.$$

Let $n \leq K_{\Omega,\mathcal{B}}$. There exists h_0, $\mathsf{s} \in \Sigma$ and $k \leq K_{\alpha_\mathsf{s}}$ such that $\mu^{[L]}_{h,\mathsf{s},k}$ is the n-th smallest element of $\mathcal{E}^L(h)$ for any $h \in (0, h_0)$. We have, by construction

$$\mu^{[L]}_{h,\mathsf{s},k} = h\mathcal{B}(\mathsf{s}) \sum_{\ell=0}^{L} h^{\ell/2} \mu^{\ell}_{\mathsf{s},k} \quad \textit{with} \quad \mu^0_{\mathsf{s},k} = \mu_k(\alpha_\mathsf{s}),$$

and there holds

$$|\mu_{h,n} - \mu^{[L]}_{h,\mathsf{s},k}| \leq C h^{\frac{L+1}{2}},\ \forall h \in (0, h_0).$$

Proof. *(i)* Using Lemma 6.9 and due to the spectral theorem (cf. [25, Chap. VII]), it follows

$$d(\sigma(P_h), \mu^{[L]}_{h,\mathsf{s},k}) \leq C\, h^{\frac{L+1}{2}}. \tag{71}$$

(ii) As in the polygonal case, we now prove a lower bound for the eigenvalues of P_h using ideas of [9, 27]. We give the sketch of the proof for the lower bound. This part is more detailed in Section 5 for polygonal case. For $\mathsf{s} \in \Sigma$, let χ_s be cut-off functions with disjoint support. We define χ_0 on Ω by $\chi_0^2 = 1 - \sum_{\mathsf{s}\in\Sigma} \chi_\mathsf{s}^2$. Due to Lemma 4.4, we know that for any $u \in H^1(\Omega)$,

$$p_h(u, u) = \sum_{\mathsf{s}\in\Sigma\cup\{0\}} p_h(\chi_\mathsf{s} u, \chi_\mathsf{s} u) - h^2 \sum_{\mathsf{s}\in\Sigma\cup\{0\}} ||u\nabla\chi_\mathsf{s}||^2_{L^2(\Omega)}. \tag{72}$$

Since $\mathrm{supp}\chi_0 \cap \Sigma = \emptyset$, we can apply the result of [17] for smooth domains which gives that there exists $c > 0$ so that

$$p_h(\chi_0 u, \chi_0 u) \geq (hb'\Theta_0 - ch^{5/4})||\chi_0 u||^2_{L^2(\Omega)}. \tag{73}$$

For any $\mathsf{s} \in \Sigma$, let T^s be the restriction of Q^{α_s} to the space spanned by the eigenfunctions $\Psi_k^{\alpha_\mathsf{s}}$ corresponding to eigenvalues $\mu_k(\alpha_\mathsf{s}) \leq \lambda_{n-1}$. We denote by $\mathcal{R}^*_{h,\mathsf{s}}$ the application

$$\begin{array}{rcll} \mathcal{R}^*_{h,\mathsf{s}} H^1(\Omega) & \to & H^1(G^{\alpha_\mathsf{s}}) & \\ \breve{u}_{h,\mathsf{s}} & \mapsto & \chi_\mathsf{s} u & \text{such that} \\ & & \breve{u}_{h,\mathsf{s}}(x) \ = \ \sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}} \exp\left(\frac{i}{h}\,\mathcal{G}_\mathsf{s}(\mathcal{R}_\mathsf{s}(x-\mathsf{s}))\right)\ u\left(\sqrt{\frac{\mathcal{B}(\mathsf{s})}{h}}\mathcal{R}_\mathsf{s}(x-\mathsf{s})\right). & \end{array}$$

Then, using Lemma 6.3 and keeping the first order term in the expansion, we have

$$\begin{aligned} p_h(\chi_{\mathsf{s}}u,\chi_{\mathsf{s}}u) &\geq h\mathcal{B}(\mathsf{s})\, q^{\alpha_{\mathsf{s}}}\big(\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u),\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u)\big)-Ch^{3/2}||\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u)||^2_{\mathcal{V}(q^{\alpha_{\mathsf{s}}})} \\ &\geq h\,\lambda_n||\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u)||^2_{L^2(G^{\alpha_{\mathsf{s}}})}-h\,\big\langle\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u),T^{\mathsf{s}}\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u)\big\rangle \\ &\quad -Ch^{3/2}||\mathcal{R}^*_{h,\mathsf{s}}(\chi_{\mathsf{s}}u)||^2_{\mathcal{V}(q^{\alpha_{\mathsf{s}}})}. \end{aligned} \tag{74}$$

Putting together (72)-(74) and considering the contribution of operator T^{s} on each corner s, there exists an operator T whose rank is not greater then $n-1$ such that

$$p_h(u,u)\geq h\lambda_n||u||^2_{L^2(\Omega)}-\langle u,Tu\rangle-ch^{5/4}||u||^2_{L^2(\Omega)}.$$

To obtain a lower bound for $\mu_{h,n}$, we use the max-min principle. Let $u_1,\dots,u_{n-1}$ belong to the orthogonal space of $\ker(T)$. Then for all ψ in $\{u_1,\dots,u_{n-1}\}^{\perp}$, ψ belongs to $\ker(T)$ and we have

$$\mu_{h,n}\geq\frac{\langle\psi,P_h\psi\rangle}{||\psi||^2}\geq h\lambda_n-ch^{5/4}. \tag{75}$$

(iii) According to (71), we know that there exists $\mu_h\in\sigma(P_h)$ such that

$$|\mu_h-h\mu^{[L]}_{h,\mathsf{s},k}|\leq Ch^{\frac{L+1}{2}}.$$

Using (75), we obtain that μ_h belongs to the set $\{\mu_{h,k},\ \lambda_k=\lambda_n\}$. □

Remark 7.2. Without the Assumption 6.2 of simplicity, the result of Theorem 7.1 is still valid but requires a more technical proof, see [10] for a construction of quasi-modes in the case when the limiting problem may have multiple eigenvalues.

7.2 Eigenspaces

Notation 7.3. • Using Notation 6.1, we denote by $\{\Lambda_1<\ldots<\Lambda_M\}$ the set of distinct values in $\{\lambda_1,\dots,\lambda_N\}$.
• For any $n\leq N$, we denote by $(\mu_{h,n},u_{h,n})$ the n-th eigenpair of P_h.
• For any $m\leq M$, we define the m-th cluster of eigenspaces of P_h by

$$F_{h,m}=\operatorname{span}\{u_{h,n}\text{ for any }n\text{ such that }\lambda_n=\Lambda_m\},$$

and the corresponding cluster of quasi-modes for any $L\in\mathbb{N}$,

$$E^{[L]}_{h,m}=\operatorname{span}\{\phi^{[L]}_{h,\mathsf{s},k}\text{ for any }\mathsf{s}\in\Sigma,k\geq1\text{ such that }\mathcal{B}(\mathsf{s})\mu_k(\alpha_{\mathsf{s}})=\Lambda_m\}.$$

By using Theorem 5.4, we obtain:

Theorem 7.4. *For any $m\leq M$ and $L\geq2$, there exists $C>0$ such that*

$$d(F_{h,m},E^{[L]}_{h,m})\leq Ch^{\frac{L-1}{2}}.$$

Proof. For any $m \leq M$, we define

$$\Sigma_m^* = \big\{(\mathsf{s}, k) \in \Sigma \times \{1, \cdots, N\}, \quad \mathcal{B}(\mathsf{s})\mu_k(\alpha_{\mathsf{s}}) = \Lambda_m\big\}.$$

(i) If the set Σ_m^* is reduced to one element, then Theorem 5.6 is obvious.

(ii) We assume that Σ_m^* is not reduced to one element. We denote by $\kappa_m^{\min}$ the smallest eigenvalue of the matrix

$$\Big(\langle \phi_{h,\mathsf{s},k}^{[L]}, \phi_{h,\mathsf{s}',k'}^{[L]} \rangle\Big)_{((\mathsf{s},k),(\mathsf{s}',k'))\in\Sigma_m^*\times\Sigma_m^*}.$$

According to Lemma 6.9, there exists C_L such that

$$|\kappa_m^{\min} - 1| \leq C_L h^{1/2}.$$

Let us now apply Theorem 5.4 with $A = P_h$. Relation (69) of Lemma 6.9 gives (45) with $\eta = Ch^{\frac{L+1}{2}}$. According to Theorem 7.1, there exists $C_1 > 0$ and $C_2 > 0$ such that, if we define $I = \big[h\mathcal{B}(\mathsf{s})\mu_k(\alpha_{\mathsf{s}}) - C_1 h^{3/2}, h\mathcal{B}(\mathsf{s})\mu_k(\alpha_{\mathsf{s}}) + C_1 h^{3/2}\big]$, for any $(\mathsf{s}, k) \in \Sigma_m^*$, we have for h small enough,

$$\sigma(P_h) \cap \big(I + B(0, 2C_2 h) \setminus I\big) = \emptyset.$$

For example, we choose $C_2 = \min\left\{\frac{\Lambda_m - \Lambda_{m-1}}{4}, \frac{\Lambda_{m+1} - \Lambda_m}{4}\right\}$ with the convention $\Lambda_0 = 0$. Assumptions of Theorem 5.4 are filled and this ends the proof of Theorem 7.4. ☐

Remark 7.5. With Notation 7.3, let $m \leq M$ and $L \geq 2$. If there exist distinct values $\mu_{h,\mathsf{s},k}^{[L]}$ such that $\mathcal{B}(\mathsf{s})\mu_k(\alpha_{\mathsf{s}}) = \Lambda_m$, we may split $E_{h,m}^{[L]}$ into several sub-clusters determined by the different values $\mu_{h,\mathsf{s},k}^{[L]}$, and, accordingly, $F_{h,m}$ splits into several sub-clusters.

8 Conclusion

As a consequence of Theorem 7.1 the smallest eigenvalue of the Schrödinger operator P_h tends to $h\lambda_1$, with λ_1 the minimum over the corners s of Ω of the characteristic quantity $\mathcal{B}(\mathsf{s})\mu_1(\alpha_{\mathsf{s}})$. Do we have a convergence of the corresponding ground state? If λ_1 is attained for one corner s only, then the ground state converges to the corresponding quasi-mode, which is exponentially localized near this corner. If λ_1 is attained for several corners, we deduce from Theorem 7.4 that the ground state is a linear combination of quasi-modes attached to these different corners. A priori, there is no reason that the coefficients in this combination converge as $h \to 0$. Moreover, a tunneling effect may happen, that is stronger interaction between some members of the cluster.

Refined information on the behavior of eigenpairs can be extracted from numerical approximation of the Schrödinger problem by finite element method, insofar such an approximation is accurate enough. We can see from our formulas, see (29) and (59), that eigenmodes have a two-scale structure: they concentrate near corners with the scale $\sqrt{h}$ but they have a strongly oscillating term of the form $\exp(i\Phi(x)/h)$ and their approximation is very delicate. We investigate a finite element method using high degree polynomials in [7]. Numerical computations do exhibit tunneling effect with multiple crossings between eigenvalues when the domain presents some symmetry (for example, in a square). We hope to propose a theoretical interpretation by analyzing an interaction matrix in a future paper.

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